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POSTULATES FOR SEPARATION OF POINT-PAIRS (REVERSIBLE ORDER ON A CLOSED LINE)

BY EDWARD V. HUNTINGTON
AND
KURT E. ROSINGER,
HARVARD UNIVERSITY.

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POSTULATES FOR SEPARATION OF POINT-PAIRS (REVERSIBLE ORDER ON A CLOSED LINE).

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POSTULATES FOR SEPARATION OF POINT-PAIRS (REVERSIBLE ORDER ON A CLOSED LINE).[1]

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Harvard University.

Introduction.

This paper extends to the tetradic relation ABCD the same detailed treatment that was given to the triadic relation ABC in two papers on Betweenness published in 1917 and 1924.[2]

The great variety of logical processes involved in the several hundred theorems contained in these papers, combined with the extremely simple nature of the fundamental concepts involved, make the whole subject of order a veritable gold-mine of illustrative material for the logician. It is hoped that the present paper, which is the joint product of a worker in mathematics and a worker in philosophy, may serve to stimulate further the growing interest in the problems of logic which stand on the borderland between these two fields. Certainly no such wealth of source-material of this particular kind has ever before been available, either in mathematics or in philosophy.

The methods here followed are the usual methods of postulational theory. Some improvements will be found, however, in the form of the two tables presented at the end of the paper; and the new sign *vs* ("versus"), meaning "conflicts with," will be found to be a convenient abbreviation. Thus ABCD . vs . ABDC is an alternative notation for ABCD . ABDC : = : 0. The sign $\supset$ means "If . . . then," the sign $\smile$ means "or" (in the sense of "at least one"), and the dot means "and."

From the purely geometrical point of view, this paper completes the postulational study of the four most important types of order:

[1] Presented to the American Mathematical Society by E. V. Huntington on April 11, 1925, and subsequently revised by the joint authors. An abstract of the paper, with special reference to Set X, was published in the Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 687–689.

[2] E. V. Huntington and J. R. Kline, Sets of independent postulates for betweenness, Transactions Amer. Math. Soc., vol. 18 (1917), pp. 301–325; and E. V. Huntington, A new set of postulates for betweenness with proof of complete independence, ibid., vol. 26 (1924), pp. 257–282. See also W. E. Van de Walle, On the complete independence of the postulates for betweenness, ibid., vol. 26 (1924), pp. 249–256.

(1) *serial order*, of points on a directed open line; (2) *cyclic order*, of points on a directed closed line;[3] (3) *betweenness*, of points on an undirected open line; (4) *separation*, of pairs of points on an undirected closed line. The theory of separation of point-pairs, or reversible order on a closed line, was first systematically studied by Pasch in 1882. The first set of postulates for this theory was given by Vailati in 1895.

Basic List of Seventeen Postulates.

In the following general list, we collect together all the seventeen postulates whose logical inter-relations we propose to study in the present paper. These postulates relate to a system (K, R) in which K is a class of at least four elements A, B, C, . . .; and R is a tetradic relation, denoted by R (ABCD), or simply ABCD.

The most familiar example of a system (K, R) in which all these postulates are true is that in which K means the class of points on a closed line, and R (ABCD) means that the pair of points AC separates the pair of points BD. In other words, these postulates characterize the theory of reversible order on a closed line.

The first postulate (Postulate D) serves to exclude obviously trivial cases and will be assumed without further mention throughout the paper. The lettered postulates, F, F′, G, H, and R, R′, concern four elements. The numbered postulates, 10–19, concern five elements. The numbering of these postulates is so chosen as to avoid confusion with the numbering already used in the papers on betweenness.

Postulate D. If ABCD, then A, B, C, D are distinct. (Distinctness.)

In all the following postulates, it is understood that distinct letters denote distinct elements of the class K.

Postulate F. If A, B, C, D are any distinct elements, then at least one of the twenty-four permutations ABCD, ABDC, . . . DCBA, will form a true tetrad. ("Foursomeness.")

Postulate F′. At least one tetrad is true, say XYZW.

Postulate G. ABCD . ⊃ . BCDA. (Cyclicity.)

Postulate H. ABCD . ABDC : = : O. ("Homogeneity.")

Postulate R. Every true tetrad is reversible: ABCD . ⊃ . DCBA. (Reversibility.)

[3] See E. V. Huntington, Sets of completely independent postulates for cyclic order, Proceedings National Academy of Sciences, vol. 10 (1924), pp. 74–78.

POSTULATE R′. At least one true tetrad is reversible.
POSTULATE 10. ABCD . X . ⊃ . AXCD ∪ ABCX. ("Connexity.")
POSTULATE 11. ABXC . ABCY . ⊃ . ABXY.
POSTULATE 12. ABXC . ABCY . ⊃ . BXCY.
POSTULATE 13. ABXC . ABCY . ⊃ . AXCY.
POSTULATE 14. ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
POSTULATE 15. ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
POSTULATE 16. ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
POSTULATE 17. ABCX . ABCY . ⊃ . (ABXY ∪ ACYX) . (ABYX ∪ ACXY).
POSTULATE 18. ABCX . ABCY . ⊃ . (ABXY ∪ BCYX) . (ABYX ∪ BCXY).
POSTULATE 19. ABCX . ABCY . ⊃ . (ACXY ∪ BCYX) . (ACYX ∪ BCXY).

The list as a whole contains, of course, many redundancies. In particular, Postulate F′ is merely a weaker form of F, and Postulate R′ is merely a weaker form of R.

In the main part of the paper, all the ways in which any one of these seventeen postulates can be deduced from any others are established, in 267 theorems on deducibility.

In the second part of the paper, 88 examples of systems (K, R) are given, which are used, in the usual way, to provide proofs of consistency and independence. These examples yield 154 theorems on non-deducibility.

As the main result of these two sets of theorems, it is found that from the complete list of seventeen postulates it is possible to select *ten and only ten sets of independent* postulates, as follows:

I. F, G, H, R, 10.	IV. F, G, H, R, 11, 14.	VII. F, G, H, R, 11, 17.
II. F, G, H, R, 12.	V. F, G, H, R, 11, 15.	VIII. F, G, H, R, 11, 18.
III. F, G, H, R, 13.	VI. F, G, H, R, 11, 16.	IX. F, G, H, R, 11, 19.
		X. F′, R′, G, H, 10.

The simplest of these sets is the last: F′, R′, G, H, 10. In this set, Postulate F′ serves merely to exclude the trivial case of a system (K, R) in which no tetrad is true. Postulates G, H, 10 apply to any system of order on a closed line, whether reversible or non-reversible. [It can be shown (see Theorem R. 1, Corollary) that in any system which satisfies G, H, 10 either every true tetrad is reversible or no true tetrad is reversible.] Postulate R′ serves to exclude the non-reversible case.[4]

[4] If Postulate R′ is replaced by the following: Postulate S′: At least one true tetrad is non-reversible; then the postulates F′, S′, G, H, 10 will form a set of independent postulates for cyclic order. See a forthcoming paper by E. V. Huntington on "The inter-relations of four types of order."

In the proofs which follow, every step of the demonstration is given explicitly, except in the case of two theorems (14.31 and 15.45) in which the proof is left to the reader. It will be observed that there are no proofs for Postulates D, G and H. The only proofs for F′ and R′ are the obvious proofs from F and R respectively.

Eight Proofs of Postulate F.

Theorem F. 1. *Proof of* F *from* F′, G, H, 10.

To prove: If A, B, C, D are distinct, then at least one of the twenty-four permutations, ABCD, ABDC, . . . , DCBA, will form a true tetrad.
By Postulate F′, there exists at least one true tetrad, XYZW.
By 10, XYZW . A . ⊃ . XAZW ⌣ XYZA. By G, AZWX ⌣ AXYZ.
If AZWX, then by 10, AZWX . Y . ⊃ . AYWX ⌣ AZWY.
If AXYZ, then by 10, AXYZ . W . ⊃ . AWYZ ⌣ AXYW.
Hence at least one of the following four cases is true:

(1) AZWX . AYWX; by G, ZWXA . WXAY.
(2) AZWX . AZWY; by G, WXAZ . ZWYA.
(3) AXYZ . AWYZ; by G, XYZA . YZAW.
(4) AXYZ . AXYW; by G, YZAX . XYWA.

Now by G, XYZW . ⊃ . YZWX.
By 10, YZWX . A . ⊃ .YAWX ⌣ YZWA. By G, AWXY ⌣ AYZW.
If AWXY, then by 10, AWXY . Z . ⊃ . AZXY ⌣ AWXZ.
If AYZW, then by 10, AYZW . X . ⊃ . AXZW ⌣ AYZX.
Hence at least one of the following four cases is true:

(1′) AYZW . AYZX; by G, ZWAY . YZWA . YZXA.
(2′) AWXY . AZXY; by G, WXYA . XYAW . XYAZ.
(3′) AWXY . AWXZ; by G, WXYA . XYAW . WXZA.
(4′) AYZW . AXZW; by G, ZWAY . YZWA . ZWAX.

But by H, Case 1 conflicts with 2′ and 3′ (since WXAY . vs . WXYA) and also with 4′ (since ZWXA . vs . ZWAX). Hence 1 . ⊃ . 1′.
By H, Case 2 conflicts with 1′ and 4′ (since ZWYA . vs . ZWAY), and also with 3′ (since WXAZ . vs . WXZA). Hence 2 . ⊃ . 2′.
By H, Case 3 conflicts with 1′ and 4′ (since YZAW . vs . YZWA), and also with 2′ (since XYZA . vs . XYAZ). Hence 3 . ⊃ . 3′.
By H, Case 4 conflicts with 2′ and 3′ (since XYWA . vs . XYAW), and also with 1′ (since YZAX . vs . YZXA). Hence 4 . ⊃ . 4′.
Therefore at least one of the following four cases is true:

(a) AYZW . AZWX . AYWX . AYZX.
(b) AZWY . AZWX . AWXY . AZXY.
(c) AWYZ . AWXZ . AWXY . AXYZ.
(d) AYZW . AXZW . AXYW . AXYZ.

Hence in any case, AYZW ∪ AZWY ∪ AWYZ.
Also, AZWX ∪ AWXZ ∪ AXZW.
Also, AYWX ∪ AWXY ∪ AXYW.
Also, AYZX ∪ AZXY ∪ AXYZ.

Introducing the notation P (A, B, C, D) to mean: "at least one of the twenty-four tetrads formed from the letters A, B, C, D is true," these results may be written:[5]

XYZW . A . ⊃ . P (A, Y, Z, W) . P (X, A, Z, W) . P (Z, Y, A, W) . P (X, Y, Z, A).

But the conclusion of this proposition clearly does not depend on the order of the letters X, Y, Z, W. Hence we have the general formula:

P (X , Y, Z, W) . A . ⊃ . P (A, Y, Z, W) . P (X, A, Z, W) . P (X, Y, A, W) . P (X, Y, Z, A).

By successive applications of this formula, we have:

XYZW . A. ⊃ . P (A, Y, Z, W).
P (A, Y, Z, W) . B . ⊃ . P (A, B, Z, W).
P (A, B, Z, W) . C . ⊃ . P (A, B, C, W).
P (A, B, C, W) . D . ⊃ . P (A, B, C, D).

Therefore, XYZW . A . B . C . D . ⊃ . P (A, B, C, D), which was to be proved.

THEOREM F. 2. *Proof of* F *from* F′, G, 10, 12.

To prove: If A, B, C, D are distinct, then at least one of the twenty-four tetrads which can be formed from the elements A, B, C, D will be true; that is, prove P (A, B, C, D).
By F′, at least one tetrad is true, say XYZW. By G, YZWX.

Case 1. Let A, B, C, D be distinct from X, Y, Z, W.
Suppose P (A, Y, Z, W) = 0; that is, suppose every tetrad formed from the four elements A, Y, Z, W is false.
By 10, YZWX . A . ⊃ . YAWX ∪ YZWA. But YZWA is false, by supposition. Hence YAWX; whence, by G, XYAW.

[5] This notation is due to Professor C. H. Langford (See Trans. Amer. Math. Soc. **26**: p. 279, 1924).

By 10, XYZW . A . ⊃ . XAZW ∪ XYZA. But if XYZA, then by 12, XYZA . XYAW . ⊃ . YZAW, which is false. Hence XAZW; whence, by G, WXAZ.

By G, XYAW . ⊃ . WXYA. By 12, WXYA . WXAZ . ⊃ . XYAZ.

By 12, XYAZ . XYZW . ⊃ . YAZW, which is false.

Hence the supposition is false; that is, P (A, Y, Z, W) is true.

Now by G, XYZW . ⊃ . YZWX . ZWXY . WXYZ, and, exactly as above, YZWX . A . ⊃ . P (A, Z, W, X) and ZWXY . A . ⊃ . P (A, W, X, Y) and WXYZ . A . ⊃ . P (A, X, Y, Z).

Hence XYZW . A . ⊃ . P (A, Y, Z, W) . P (X, A, Z, W) . P (X, Y, A, W) . P (X, Y, Z, A).

But the conclusion of this proposition clearly does not depend on the order of the letters X, Y, Z, W, so that in the premiss XYZW might have been any other tetrad formed from the letters X, Y, Z, W. Hence we have the general formula:

P (X, Y, Z, W) . A : ⊃ : P (A, Y, Z, W) . P (X, A, Z, W) . P (X, Y, A, W) . P (X, Y, Z, A).

By applying the second, third, and fourth part of this formula, we have

P (A, Y, Z, W) . B . ⊃ . P (A, B, Z, W),
P (A, B, Z, W) . C . ⊃ . P (A, B, C, W),
P (A, B, C, W) . D . ⊃ . P (A, B, C, D).

Therefore P (A, B, C, D), which was to be proved.

Case 2. If any of the elements A, B, C, D are equal to any of the elements X, Y, Z, W, certain steps of the above proof are unnecessary. Hence in any case, P (A, B, C, D).

THEOREM F. 3. *Proof of* F *from* F′, G, 10, 13.

To prove: If A, B, C, D are distinct, then at least one of the twenty-four tetrads which can be formed from the elements A, B, C, D will be true; that is, prove P (A, B, C, D).

By F′, at least one tetrad is true, say XYZW. By G, YZWX.

Case 1. Let A, B, C, D be distinct from XYZW.

Suppose P (A, Y, Z, W) = O; that is, suppose every tetrad formed from the four elements A, Y, Z, W is false.

By 10, YZWX . A . ⊃ . YAWX ∪ YZWA. But YZWA is false. Hence YAWX; whence by G, XYAW and WXYA.

Now by G, YZWX . ⊃ . ZWXY. By 10, ZWXY . A . ⊃ . ZAXY ∪ ZWXA.

But if ZWXA, then by G, WXAZ, and by 13, WXYA . WXAZ . ⊃ . WYAZ, which is false.
Hence ZAXY, whence by G, XYZA. By 13, XYZA . XYAW . ⊃ . XZAW, whence, by G, WXZA.
Now by G, XYZW . ⊃ . WXYZ. By 13, WXYZ . WXZA . ⊃ . WYZA, which is false.
Hence the supposition is false; that is, P (A, Y, Z, W) is true.
From this point, the proof (including Case 2) duplicates the proof of Theorem F. 2.
Therefore P (A, B, C, D).

THEOREM F. 4. *Proof of* F *from* F′, G, 10, 14.

To prove: If A, B, C, D are distinct, then P (A, B, C, D).
By F′, XYZW. By G, YZWX . ZWXY . WXYZ.

Case 1. Let A, B, C, D be distinct from X, Y, Z, W.
Suppose P (A, Y, Z, W) = 0.
By 10, YZWX . A . ⊃ . YAWX ∪ YZWA. But YZWA is false. Hence YAWX; whence by G, WXYA and AWXY.
By 14, WXYA . WXYZ . ⊃ . WXAZ ∪ WXZA.
If WXAZ, then by G, ZWXA, whence by 14,
ZWXA . ZWXY . ⊃ . ZWAY ∪ ZWYA, both of which are false.
If WXZA, then by G, AWXZ, whence by 14,
AWXZ . AWXY . ⊃ . AWZY ∪ AWYZ, both of which are false.
Hence the supposition is false; that is P (A, Y, Z, W) is true.
From this point, the proof duplicates the proof of Theorem F. 2.
Therefore P (A, B, C, D).

THEOREM F. 5. *Proof of* F *from* F′, G, 10, 15.

To prove: If A, B, C, D are distinct, then P (A, B, C, D).
By F′, XYZW. By G, ZWXY.

Case 1. Let A, B, C, D be distinct from X, Y, Z, W.
Suppose P (A, Y, Z, W) = 0.
By 10, ZWXY . A . ⊃ . ZAXY ∪ ZWXA.
If ZAXY, then by G, AXYZ, whence by 10, AWYZ ∪ AXYW. But AWYZ is false, and if AXYW, then by 15, AXYW . AXYZ . ⊃ . AYWZ ∪ AYZW, both of which are false.
If ZWXA, then by G, WXAZ, whence by 10, WYAZ ∪ WXAY. But WYAZ is false, and if WXAY, then by 15, WXAY . WXAZ . ⊃ . WAYZ ∪ WAZY, both of which are false.

Hence the supposition is false; that is, P (A, Y, Z, W) is true.
From this point, the proof duplicates the proof of Theorem F. 2.
Therefore P (A, B, C, D).

THEOREM F. 6. *Proof of* F *from* F′, G, 10, 16.

To prove: If A, B, C, D are distinct, then P (A, B, C, D).
By F′, XYZW. By G, YZWX and WXYZ.

Case 1. Let A, B, C, D be distinct from X, Y, Z, W.
Suppose P (A, Y, Z, W) = O.
By 10, YZWX . A . ⊃ . YAWX ∪ YZWA. But YZWA is false. Hence YAWX; whence, by G, WXYA and XYAW.
By 16, WXYA . WXYZ . ⊃ . XYAZ ∪ XYZA. But if XYZA, then by 16 XYZA . XYZW . ⊃ . YZAW ∪ YZWA, both of which are false. Hence XYAZ.
By 16, XYAW . XYAZ . ⊃ . YAWZ ∪ YAZW, both of which are false.
Hence the supposition is false; that is, P (A, Y, Z, W) is true.
From this point the proof duplicates the proof of Theorem F. 2.
Therefore P (A, B, C, D).

THEOREM F. 7. *Proof of* F *from* F′, G, 10, 11, 17.

Proof: By Theorem 14. 15, Postulates G, 11, 17 give 14; and by Theorem F. 4, Postulates F′, G, 10, 14 give F.

THEOREM F. 8. *Proof of* F *from* F′, G, 10, 11, 19.

Proof: By Theorem 16. 15, Postulates G, 11, 19 give 16; and by Theorem F. 6, Postulates F′, G, 10, 16 give F.

ONE PROOF OF POSTULATE R.

THEOREM R. 1. *Proof of* R *from* G, H, 10, R′.

To prove: ABCD . ⊃ . DCBA; that is, that every true tetrad is reversible.
If Postulate F′ is false, then there is no true tetrad in the system, and Postulate R is satisfied.
If Postulate F′ is true, then by Postulate R′, we know that there is at least one true tetrad which is reversible, or that we have, say, XYZW and WZYX.
Combining XYZW with the element A, as in the proof of Theorem F, we know that at least one of the four cases, a, b, c, d (mentioned in that proof) is true. Using G, and re-arranging the terms, we may rewrite these four cases as follows:

(a') YZXA . YWXA . ZWXA . YZWA; by G, AYWX . XAYW.
(b') ZXYA . WXYA . ZWXA . ZWYA; by G, AZXY . YAZX.
(c') XYZA . WXYA . WXZA . WYZA; by G, AWYZ . ZAWY.
(d') XYZA . XYWA . XZWA . YZWA; by G, AXZW . WAXZ.

By combining WZYX with the element A in a similar way (simply replacing X by W, and Y by Z, and Z by Y, and W by X, in cases c, b, a, d, in that order) we see that at least one of the following four cases is true:

(a'') AXZY . AXWY . AXWZ . AWZY; by G, YAXZ.
(b'') AYXZ . AYXW . AXWZ . AYWZ; by G, ZAYW.
(c'') AZYX . AYXW . AZXW . AZYW; by G, WAZX.
(d'') AZYX . AWYX . AWZX . AWZY; by G, XAWY.

But by H, a' conflicts with b'' and c'' (since AYWX . vs . AYXW), and also with d'' (since XAYW . vs . XAWY). Hence $a' . \supset . a''$.
By H, b' conflicts with c'' and d'' (since AZXY . vs . AZYX), and also with a'' (since YAZX . vs . YAXZ). Hence $b' . \supset . b''$.
By H, c' conflicts with a'' and d'' (since AWYZ . vs . AWZY), and also with b'' (since ZAWY . vs . ZAYW). Hence $c' . \supset . c''$.
By H, d' conflicts with a'' and b'' (since AXZW . vs . AXWZ), and also with c'' (since WAXZ . vs . WAZX). Hence $d' . \supset . d''$.
Therefore at least one of the following four cases is true:

(I) AXZY . YZXA . AXWY . YWXA . AXWZ . ZWXA . AWZY . YZWA.
(II) AYXZ . ZXYA . AYXW . WXYA . AXWZ . ZWXA . AYWZ . ZWYA.
(III) AZYX . XYZA . AYXW . WXYA . AZXW . WXZA . AZYW . WYZA.
(IV) AZYX . XYZA . AWYX . XYWA . AWZX . XZWA . AWZY . YZWA.

Hence AXZY . YZXA : $\smile$: AYXZ . ZXYA : $\smile$: AZYX . XYZA.
Also AXWY . YWXA : $\smile$: AYXW . WXYA : $\smile$: AWYX . XYWA.
Also AXWZ . ZWXA : $\smile$: AZXW . WXZA : $\smile$: AWZX . XZWA.
Also AWZY . YZWA : $\smile$: AYWZ . ZWYA : $\smile$: AZYW . WYZA.
Introducing the notation Q (A, B, C, D) to mean "at least one tetrad formed from the letters A, B, C, D, is true *and reversible*," these results may be written:

XYZW . WZYX . A : $\supset$: Q (A, Y, Z, W) . Q (X, A, Z, W) . Q (X, Y, A, W) . Q (X, Y, Z, A).

The conclusion of this proposition is clearly not affected by changing the order of the letters X, Y, Z, W. Hence we have the general formula:

Q (X, Y, Z, W) . A : ⊃ : Q (A, Y, Z, W) . Q (X, A, Z, W) . Q (X, Y, A, W) . Q (X, Y, Z, A).

By successive applications of this general formula we have:

XYZW . WZYX . A : ⊃ : Q (A, Y, Z, W);
then Q (A, Y, Z, W) . ⊃ . Q (A, B, Z, W);
then Q (A, B, Z, W) . ⊃ . Q (A, B, C, W);
then Q (A, B, C, W) . ⊃ . Q (A, B, C, D).
That is, XYZW . WZYX . A : ⊃ : Q (A, B, C, D).

This result tells not only that at least one tetrad formed from the letters A, B, C, D, is true (which we knew already from the hypothesis of the theorem), but also that at least one such true tetrad is reversible.

Now the twenty-four tetrads which can be formed from the letters A, B, C, D may be arranged in six groups, as follows:

(1) ABCD, BCDA, CDAB, DABC;
(2) ABDC, BDCA, DCAB, CABD;
(3) ACBD, CBDA, BDAC, DACB;
(4) CBAD, BADC, ADCB, DCBA;
(5) DBAC, BACD, ACDB, CDBA;
(6) BCAD, CADB, ADBC, DBCA.

By G, all tetrads in group (1) are true (since ABCD is true); and by G and H, all in groups (2) and (3) are false. Suppose group (4) is false; then the only tetrads which can be true will be found in group (1), (5), or (6). Since at least one of these is reversible, at least one tetrad in groups (4), (2), or (3) must be true, contrary to the supposition. Hence group (4) must be true.

Therefore DCBA.

Corollary. In any system which satisfies G, H, and 10, either every true tetrad is reversible or else no true tetrad is reversible.

For, in such a system, if R′ is true then R is true (by Theorem R. 1); that is, every true tetrad is reversible; and if R′ is false, then no true tetrad is reversible.

Eight Proofs of Postulate 10.

Theorem 10. 1. *Proof of* 10 *from* R, F, G, 12.

To prove: ABCD . X . ⊃ . AXCD ∨ ABCX.

By F, at least one of the twenty-four permutations of the elements A, X, C, D will form a true tetrad.

These twenty-four permutations may be arranged in three lines, as follows:

AXCD, XCDA, CDAX, DAXC; CXAD, XADC, ADCX, DCXA.
AXDC, XDCA, DCAX, CAXD; DXAC, XACD, ACDX, CDXA.
ACXD, CXDA, XDAC, DACX; XCAD, CADX, ADXC, DXCA.

By R, G, if any tetrad in the first line is true, then AXCD.
By R, G, if any tetrad in the second line is true, then AXDC.
By R, G, if any tetrad in the third line is true, then ACXD.
Hence, by F, R, G, we have AXCD ⌣ AXDC ⌣ ACXD.
Suppose AXCD = 0. Then AXDC ⌣ ACXD. By G, DCAX ⌣ DACX.

Case 1. Suppose DCAX. By R, ABCD . ⊃ . DCBA.
By 12, DCBA . DCAX . ⊃ . CBAX. By R, G, ABCX.

Case 2. Suppose DACX. By G, ABCD . ⊃ . DABC.
By 12, DABC . DACX . ⊃ . ABCX.
Therefore AXCD ⌣ ABCX.

THEOREM 10. 2. *Proof of* 10 *from* R, F, G, 13.

To prove: ABCD . X . ⊃ . AXCD ⌣ ABCX.
By F, R, G (as in the preceding theorem), AXCD ⌣ AXDC ⌣ ACXD.
By G, ABCD . ⊃ . CDAB. By R, G, ABCD . ⊃ . ADCB.
Suppose AXCD = 0. Then AXDC ⌣ ACXD. By R, G, CDXA ⌣ ADXC.

Case 1. If CDXA, by13, CDXA . CDAB . ⊃ . CXAB. By G, ABCX.

Case 2. If ADXC, by 13, ADXC . ADCB . ⊃ . AXCB. By R, G, ABCX.
Therefore AXCD ⌣ ABCX.

THEOREM 10. 3. *Proof of* 10 *from* R, F, G, 11, 14.

Proof: Postulate 15 follows from R, G, 11, 14, by Theorem 15. 25, (below); then 10 follows from R, F, G, 11, 15, by Theorem 10. 4 (below).

THEOREM 10. 4. *Proof of* 10 *from* R, F, G, 11, 15.

To prove: ABCD . X . ⊃ . AXCD ⌣ ABCX.
By R, G, ABCD . ⊃ . ADCB. By R, G, ABCD . ⊃ . BADC.
By R, ABCD . ⊃ . DCBA. By G, ABCD . ⊃ . DABC.
By G, ABCD . ⊃ . CDAB. By G, ABCD . ⊃ . BCDA.
By F, R, G (as in Theorem 10. 1), AXCD ⌣ AXDC ⌣ ACXD.
Suppose AXCD = 0. Then AXDC ⌣ ACXD. By G, DCAX ⌣ DACX.

Case 1. Suppose DCAX. By 11, DCBA . DCAX . ⊃ . DCBX.
By 15, DCBX . DCBA . ⊃ . DBXA ⌣ DBAX. By G, ADBX ⌣ BAXD.

If ADBX, by 11, ADCB . ADBX . ⊃ . ADCX; by R, G, AXCD.
If BAXD, by 11, BAXD . BADC . ⊃ . BAXC; by R, G, ABCX.
Hence in Case 1, AXCD ∪ ABCX.

Case 2. Suppose DACX. By 11, DABC . DACX . ⊃ . DABX.
By 15, DABX . DABC . ⊃ . DBXC ∪ DBCX. By G, CDBX ∪ BCXD.
If CDBX, by 11, CDAB . CDBX . ⊃ . CDAX; by G, AXCD.
If BCXD, by 11, BCXD . BCDA . ⊃ . BCXA; by G, ABCX.
Hence in Case 2, AXCD ∪ ABCX.
Therefore AXCD ∪ ABCX.

THEOREM 10. 5. *Proof of* 10 *from* R, F, G, 11, 16.

This theorem follows from Theorem 14. 20 (below) by aid of Theorem 10. 3.

THEOREM 10. 6. *Proof of* 10 *from* R, F, G, 11, 17.

This theorem follows from Theorem 15. 27 (below), by aid of Theorem 10. 4.

THEOREM 10. 7. *Proof of* 10 *from* R, F, G, 11, 18.

To prove: ABCD . X . ⊃ . AXCD ∪ ABCX.
By R, ABCD . ⊃ . DCBA. By G, ABCD . ⊃ . DABC.
By F, R, G (as in Theorem 10. 1), AXCD ∪ AXDC ∪ ACXD.
Suppose AXCD = 0. Then AXDC ∪ ACXD. By G, DCAX ∪ DACX.

Case 1. If DCAX, by 11, DCBA . DCAX . ⊃ . DCBX.
By 18, DCBX . DCBA . ⊃ . DCXA ∪ CBAX. By R, G, AXCD ∪ ABCX.

Case 2. If DACX, by 11, DABC . DACX . ⊃ . DABX.
By 18, DABX . DABC . ⊃ . DAXC ∪ ABCX. By G, AXCD ∪ ABCX.
Therefore AXCD ∪ ABCX.

THEOREM 10. 8. *Proof of* 10 *from* R, F, G, 11, 19.

This follows from Theorem 15. 28 (below), by aid of Theorem 10. 4.

SIX PROOFS OF POSTULATE 11.

THEOREM 11. 1. *Proof of* 11 *from* G, H, 10.

To prove: ABXC . ABCY . ⊃ . ABXY.
By 10, ABCY . X . ⊃ . AXCY ∪ ABCX. But ABCX . VS . ABXC, by H.
Hence AXCY, whence, by G, XCYA.

By 10, ABXC . Y . ⊃ . AYXC ∪ ABXY.
But if AYXC, then by G, XCAY, and XCAY . vs . XCYA, by H.
Therefore ABXY.

THEOREM 11. 2. *Proof of* 11 *from* G, 12.

To prove: ABXC . ABCY . ⊃ . ABXY.
By 12, ABXC . ABCY . ⊃ . BXCY, whence by G, CYBX.
By G, ABCY . ⊃ . CYAB.
By 12, CYAB . CYBX . ⊃ . YABX. Hence by G, ABXY.

THEOREM 11. 3. *Proof of* 11 *from* G, 13.

To prove: ABXC . ABCY . ⊃ . ABXY.
By 13, ABXC . ABCY . ⊃ . AXCY, whence by G, XCYA.
By G, ABXC . ⊃ . XCAB.
By 13, XCYA . XCAB . ⊃ . XYAB. Hence by G, ABXY.

THEOREM 11. 4. *Proof of* 11 *from* R, 12, 13.

To prove: ABXC . ABCY . ⊃ . ABXY.
By 12, ABXC . ABCY . ⊃ . BXCY, whence by R, YCXB.
By R, ABCY . ⊃ . YCBA.
By 13, YCXB . YCBA . ⊃ . YXBA, whence by R, ABXY.

THEOREM 11. 5. *Proof of* 11 *from* R, H, 10.

To prove: ABXC . ABCY . ⊃ . ABXY.
Suppose ABXY = o. Then by R, YXBA = o.
By 10, ABXC . Y . ⊃ . AYXC ∪ ABXY. But ABXY = o. Hence AYXC.
By R, ABXC . ⊃ . CXBA and by R, ABCY . ⊃ . YCBA.
By 10, ABCY . X . ⊃ . AXCY ∪ ABCX. But ABCX . vs . ABXC by H.
Hence AXCY. By R, YCXA.
By 10, CXBA . Y. ⊃ . CYBA ∪ CXBY, whence by R, ABYC ∪ YBXC.
But ABYC . vs . ABCY by H. Hence YBXC.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. By R, ABXY ∪ XBCY.
But ABXY = o. Hence XBCY.
By 10, XBCY . A . ⊃ . XACY ∪ XBCA. By R, YCAX ∪ ACBX.
But YCAX . vs . YCXA by H. Hence ACBX.
By 10, ACBX . Y . ⊃ . AYBX ∪ ACBY. By R, XBYA ∪ YBCA.
Suppose XBYA. Then by 10, XBYA . C . ⊃ . XCYA ∪ XBYC, whence by R, AYCX ∪ XBYC.
But AYCX . vs . AYXC by H, and XBYC . vs . XBCY by H.

Therefore YBCA.
By 10, YBCA . X . ⊃ . YXCA ∪ YBCX. But YBCX . vs . YBXC by H.
Hence YXCA. By R, ACXY.
By 10, ACXY . B . ⊃ . ABXY ∪ ACXB. But ACXB . vs . ACBX by H.
Therefore ABXY.

THEOREM 11. 6. *Proof of* 11 *from* R, 10, 12, 17.

To prove: ABXC . ABCY . ⊃ . ABXY.
Suppose ABXY = 0. By R, YXBA = 0.
By 10, ABXC . Y . ⊃ . AYXC ∪ ABXY. Hence AYXC. By R, CXYA.
By R, ABXC . ⊃ . CXBA. By R, ABCY . ⊃ . YCBA.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. Hence YCBX. By R, XBCY.
By 12, ABXC . ABCY . ⊃ . BXCY. By R, YCXB.

Case 1. Suppose YBAX = 0. By R, XABY = 0.
By 17, YCBX . YCBA . ⊃ . YCXA ∪ YBAX. Hence YCXA. By R, AXCY.
By 17, YCXA . YCXB . ⊃ . YCAB ∪ YXBA. Hence YCAB. By R, BACY.
By 12, YCXA . YCAB . ⊃ . CXAB.
By 12, YCAB . YCBX . ⊃ . CABX.
Now CAXY = 0 (for if not, by 12, CABX . CAXY . ⊃ . ABXY). By R, YXAC = 0.
Also, CXBY = 0 (for if not, by 12, CXAB . CXBY . ⊃ . XABY). By R, YBXC = 0.
Then AYBX = 0 (for if not, by 12, AYBX . AYXC . ⊃ . YBXC). By R, XBYA = 0.
By 10, CXBA . Y . ⊃ . CYBA ∪ CXBY. Hence CYBA.
Now CYAX = 0 (for if not, by 12, CYBA . CYAX . ⊃ . YBAX). By R, XAYC = 0.
Also YBAC = 0 (for if not, by 10, YBAC . X . ⊃ . YXAC ∪ YBAX, while YXAC = 0 and YBAX = 0).
Also XBYC = 0 (for if not, by 10, XBYC . A . ⊃ . XAYC ∪ XBYA, while XAYC = 0 and XBYA = 0).
By 10, CABX . Y . ⊃ . CYBX ∪ CABY. By R, XBYC ∪ YBAC.
But XBYC = 0 and YBAC = 0. Hence the supposition in case 1 is impossible.
Therefore YBAX. By R, XABY.

Case 2. Suppose YCAX = 0. From Case 1, YBAX.
By 17, YCBA . YCBX . ⊃ . YCAX ∪ YBXA. Hence YBXA.
By 10, YBAX . C . ⊃ . YCAX ∪ YBAC. Hence YBAC.
By 12, YBXA . YBAC . ⊃ . BXAC.

By 12, BXAC . BXCY . ⊃ . XACY. By R, YCAX. But YCAX = 0.
Hence the supposition in case 2 is impossible.
Therefore YCAX . By R, XACY.
By 12, YCAX . YCXB . ⊃ . CAXB. By R, BXAC.
By 12, YCBA . YCAX . ⊃ . CBAX.

Case 3. Suppose CABX. From Case 1, YBAX. From Case 2, YCAX.
Now CAXY = 0 (for if not, by 12, CABX . CAXY . ⊃ . ABXY).
By 10, CAXB . Y . ⊃ . CYXB ∪ CAXY. Hence CYXB. By R, BXYC.
Now CYBA = 0 (for if not, by 12, CYXB . CYBA . ⊃ . YXBA). By R, ABYC = 0.
Then XAYC = 0 (for if not, by 12, XABY . XAYC . ⊃ . ABYC). By R, CYAX = 0.
Then BXAY = 0 (for if not, by 12, BXAY . BXYC . ⊃ . XAYC). By R, YAXB = 0.
By 10, CBAX . Y . ⊃ . CYAX ∪ CBAY. Hence CBAY.
Now CBXA = 0 (for if not, by 12, CBXA . CBAY . ⊃ . BXAY).
Then YCXA = 0 (for if not, by 12, YCBX . YCXA . ⊃ . CBXA).
By 10, YCXB . A . ⊃ . YAXB ∪ YCXA. But YAXB = 0 and YCXA = 0.
Therefore CABX = 0. By R, XBAC = 0.
Now YCAB = 0 (for if not, by 12, YCAB . YCBX . ⊃ . CABX). By R, BACY = 0.
By 10, YCXB . A . ⊃ . YAXB ∪ YCXA.
But if YCXA, then by 17, YCXA . YCXB . ⊃ . YCAB ∪ YXBA, where YCAB = 0 and YXBA = 0. Hence YAXB. By R, BXAY.

Case 4. Suppose CYXB. By R, BXYC.
Now CYBA = 0 (for if not, by 12, CYXB . CYBA . ⊃ . YXBA). By R, ABYC = 0.
By 12, BXAY . BXYC . ⊃ . XAYC. From Case 1, XABY.
By 12, XABY . XAYC . ⊃ . ABYC. But ABYC = 0.
Hence the supposition in Case 4 is impossible.
Therefore CYXB = 0. By R, BXYC = 0.
By 17, BXAY . BXAC . ⊃ . BXYC ∪ BACY. Hence BACY. But BACY = 0.
Hence the supposition that ABXY = 0 is impossible.
Therefore ABXY.

Thirteen Proofs of the Postulate 12.

THEOREM 12. 1. *Proof of* 12 *from* G, H, 10.

To prove: ABXC . ABCY . ⊃ . BXCY.
By G, ABXC . ⊃ . BXCA.

By 10, BXCA . Y . ⊃ . BYCA ∪ BXCY. But if BYCA, then by G, ABYC, which conflicts with ABCY, by H.
Therefore BXCY.

THEOREM 12. 2. *Proof of* 12 *from* G, 13.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 13, ABXC . ABCY . ⊃ . AXCY, whence by G, XCYA.
By G, ABXC . ⊃ . XCAB. By G, XCYA . ⊃ . YAXC.
By 13, XCYA . XCAB . ⊃ . XYAB, whence by G, YABX.
By 13, YABX . YAXC . ⊃ . YBXC, whence by G, BXCY.

THEOREM 12. 3. *Proof of* 12 *from* G, H, 11, 15.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY, whence by G, YABX.
By G, ABCY . ⊃ . YABC. By G, ABCY . ⊃ . BCYA.
By 15, YABX . YABC . ⊃ . YBXC ∪ YBCX, whence by G, BXCY ∪ BCXY.
But if BCXY, then by 11, BCXY . BCYA . ⊃ . BCXA, whence by G, ABCX, which conflicts with ABXC, by H.
Therefore BXCY.

THEOREM 12. 4. *Proof of* 12 *from* G, H, 11, 17.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY, whence by G, YABX.
By 17, ABXY . ABXC . ⊃ . ABYC ∪ AXCY. But ABYC . VS . ABCY, by H.
Therefore AXCY, whence by G, YAXC. By G. ABCY . ⊃ . YABC.
By 17, YABC . YABX . ⊃ . YACX ∪ YBXC. But YACX . VS . YAXC, by H.
Therefore YBXC, whence by G, BXCY.

THEOREM 12. 5. *Proof of* 12 *from* H, 11, 18.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 18, ABXY . ABXC . ⊃ . ABYC ∪ BXCY.
But ABYC . VS . ABCY, by H. Therefore BXCY.

THEOREM 12. 6. *Proof of* 12 *from* G, H, 11, 19.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY, whence by G, YABX.
By G, ABCY . ⊃ . YABC.
By 19, YABX . YABC . ⊃ . YBXC ∪ ABCX. But ABCX . VS . ABXC, by H.
Therefore YBXC, whence by G, BXCY.

THEOREM 12. 7. *Proof of* 12 *from* H, 10, 11, 19.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 10, ABCY . X . ⊃ . AXCY ∨ ABCX.
But ABCX . vs . ABXC, by H. Therefore AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 19, ABXY . ABXC . ⊃ . AXYC ∨ BXCY.
But AXYC . vs . AXCY. Therefore BXCY.

THEOREM 12. 8. *Proof of* 12 *from* H, 11, 13, 19.
To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 13, ABXC . ABCY . ⊃ . AXCY.
By 19, ABXY . ABXC . ⊃ . AXYC ∨ BXCY.
But AXYC . vs . AXCY, by H. Therefore BXCY.

THEOREM 12. 9. *Proof of* 12 *from* H, 11, 17, 19.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 17, ABXY . ABXC . ⊃ . ABYC ∨ AXCY.
But ABYC . vs . ABCY, by H. Therefore AXCY.
By 19, ABXY . ABXC . ⊃ . AXYC ∨ BXCY.
But AXYC . vs . AXCY, by H. Therefore BXCY.

THEOREM 12. 10. *Proof of* 12 *from* R, G, H, 11, 14.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY, whence by R, G, XBAY.
By R, G, ABXC . ⊃ . XBAC.
By 14, XBAC . XBAY . ⊃ . XBCY ∨ XBYC, whence by R, G, BCYX ∨ BXCY.
Suppose BCYX. By G, ABCY . BCYA. By G, ABXC . ⊃ . XCAB.
By 14, BCYA . BCYX . ⊃ . BCAX ∨ BCXA, whence by R, G, XBCA ∨ XCBA.
But XBCA . vs . XBAC, by H; and XCBA . vs . XCAB, by H.
Hence BCYX is false. Therefore BXCY.

THEOREM 12. 11. *Proof of* 12 *from* R, G, H, 11, 16.

To prove: ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY. By G, ABXC . ⊃ . BXCA.
By 16, ABXY . ABXC . ⊃ . BXYC ∨ BXCY.
Suppose BXYC. Then by G, YCBX. By R, ABCY . ⊃ . YCBA.
By 16, YCBA . YCBX . ⊃ . CBAX ∨ CBXA, whence by R, G, ABCX ∨ BXAC.

But ABCX . vs . ABXC, by H; and BXAC . vs . BXCA, by H.
Hence BXYC is false. Therefore BXCY.

THEOREM 12. 12. *Proof of* 12 *from* R, H, 10, 18.

Postulate 11 follows from R, H, 10, by Theorem 11. 5.
Postulate 12 follows from H, 11, 18, by Theorem 12. 5.
Therefore 12 follows from R, H, 10, 18.

THEOREM 12. 13. *Proof of* 12 *from* R, H, 10, 19.

Postulate 11 follows from R, H, 10 by Theorem 11. 5.
Postulate 12 follows from H, 10, 11, 19, by Theorem 12. 7.
Therefore 12 follows from R, H, 10, 19.

THIRTEEN PROOFS OF POSTULATE 13.

THEOREM 13. 1. *Proof of* 13 *from* H, 10.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 10, ABCY . X . ⊃ . AXCY ∪ ABCX. But ABCX . vs . ABXC, by H.
Therefore AXCY.

THEOREM 13. 2. *Proof of* 13 *from* G, 12.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 12, ABXC . ABCY . ⊃ . BXCY, whence by G, CYBX.
By G, ABCY . ⊃ . CYAB.
By 12, CYAB . CYBX . ⊃ . YABX, whence by G, BXYA.
By 12, BXCY . BXYA . ⊃ . XCYA, whence by G, AXCY.

THEOREM 13. 3. *Proof of* 13 *from* G, H, 11, 15.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By G, ABXC . ⊃ . CABX.
By 15, ABXY . ABXC . ⊃ . AXYC ∪ AXCY, whence by G, CAXY ∪ AXCY.
But if CAXY, then by 11, CABX . CAXY . ⊃ . CABY, whence by G, ABYC, which conflicts with ABCY, by H.
Therefore AXCY.

THEOREM 13. 4. *Proof of* 13 *from* H, 11, 17.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.

By 17, ABXY . ABXC . ⊃ . ABYC ∪ AXCY.
But ABYC . vs . ABCY, by H.
Therefore AXCY.

THEOREM 13. 5. *Proof of* 13 *from* G, H, 11, 18.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY, whence by G, YABX.
By G, ABCY . ⊃ . YABC.
By 18, YABX . YABC . ⊃ . YAXC ∪ ABCX. But ABCX . vs . ABXC, by H.
Therefore YAXC, whence by G, AXCY.

THEOREM 13. 6. *Proof of* 13 *from* G, H, 11, 19.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY, whence by G, YABX.
By G, ABCY . ⊃ . YABC.
By 19, YABX . YABC . ⊃ . YBXC ∪ ABCX. But ABCX . vs . ABXC, by H.
Therefore YBXC, whence by G, BXCY.
By 19, ABXC . ABXY . ⊃ . AXCY ∪ BXYC. But BXYC . vs . BXCY, by H.
Therefore AXCY.

THEOREM 13. 7. *Proof of* 13 *from* H, 11, 12, 19.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 12, ABXC . ABCY . ⊃ . BXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 19, ABXC . ABXY . ⊃ . AXCY ∪ BXYC.
But BXYC . vs . BXCY, by H.
Therefore AXCY.

THEOREM 13. 8. *Proof of* 13 *from* H, 11, 18, 19.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 18, ABXY . ABXC . ⊃ . ABYC ∪ BXCY.
But ABYC . vs . ABCY, by H. Therefore BXCY.
By 19, ABXC . ABXY . ⊃ . AXCY ∪ BXYC.
But BXYC . vs . BXCY, by H.
Therefore AXCY.

THEOREM 13. 9. *Proof of* 13 *from* R, G, H, 11, 14.

Postulate 12 follows from R, G, H, 11, 14, by Theorem 12. 10.
Postulate 13 follows from G, 12, by Theorem 13. 2.
Therefore 13 follows from R, H, G, 11, 14.

THEOREM 13. 10. *Proof of* 13 *from* R, G, H, 11, 16.

Postulate 12 follows from R, G, H, 11, 16, by Theorem 12. 11.
Postulate 13 follows from G, 12, by Theorem 13. 2.
Therefore 13 follows from R, G, H, 11, 16.

THEOREM 13. 11. *Proof of* 13 *from* R, H, 11, 18.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 11, ABXC . ABCY . ⊃ . ABXY.
By 18, ABXY . ABXC . ⊃ . ABYC ∪ BXCY. But ABYC . vs . ABCY, by H.
Therefore BXCY, whence by R, YCXB. By R, ABCY . ⊃ . YCBA.
By 11, YCXB . YCBA . ⊃ . YCXA, whence by R, AXCY.

THEOREM 13. 12. *Proof of* 13 *from* R, 11, 12.

To prove: ABXC . ABCY . ⊃ . AXCY.
By 12, ABXC . ABCY . ⊃ . BXCY, whence by R, YCXB.
By R, ABCY . ⊃ . YCBA.
By 11, YCXB . YCBA . ⊃ . YCXA, whence by R, AXCY.

THEOREM 13. 13. *Proof of* 13 *from* R, 10, 12, 17.

Postulate 11 follows from R, 10, 12, 17, by Theorem 11. 6.
Postulate 13 follows from R, 11, 12, by Theorem 13. 12.
Therefore 13 follows from R, 10, 12, 17.

THIRTY-TWO PROOFS OF POSTULATE 14.

THEOREM 14. 1. *Proof of* 14 *from* G, H, 10.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABXY = 0 and ABYX = 0. By G, BXYA = 0 and BYXA = 0.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . BCYA.
By 10, BCXA . Y . ⊃ . BYXA ∪ BCXY. Therefore BCXY.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX. Therefore BCYX.
But BCXY . vs . BCYX, by H. Therefore ABXY ∪ ABYX.

THEOREM 14. 2. *Proof of* 14 *from* G, 12, 10.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABXY is false, whence by G, BXYA is false.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . BCXA.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX. But BXYA = 0. Therefore BCYX.
By 12, BCYX . BCXA . ⊃ . CYXA. By G, ABCY . ⊃ . CYAB.

By 12, CYXA . CYAB . ⊃ . YXAB, whence by G, ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 3. *Proof of* 14 *from* G, 12, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By G, ABCX . ⊃ . CXAB. By G, ABCY . ⊃ . CYAB.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX, whence by G, CXYA ∪ CYXA.
If CXYA, then by 12, CXYA . CXAB . ⊃ . XYAB, whence by G, ABXY.
If CYXA, then by 12, CYXA . CYAB . ⊃ . YXAB, whence by G, ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 4. *Proof of* 14 *from* G, 12, 16.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . BCXA.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX.

Case 1. If BCXY, then by 12, BCXY . BCYA . ⊃ . CXYA.
By G, ABCX . ⊃ . CXAB.
By 12, CXYA . CXAB . ⊃ . XYAB, whence by G, ABXY.

Case 2. If BCYX, then by 12, BCYX . BCXA . ⊃ . CYXA.
By G, ABCY . ⊃ . CYAB.
By 12, CYXA . CYAB . ⊃ . YXAB, whence G, ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 5. *Proof of* 14 *from* G, 12, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABYX is false.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. Therefore ACXY.
By G, ACXY . ⊃ . CXYA. By G, ABCX . ⊃ . CXAB.
By 12, CXYA . CXAB . ⊃ . XYAB, whence by G, ABXY.
Therefore ABXY ∪ ABYX.

THEOREM 14. 6. *Proof of* 14 *from* G, 12, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABXY is false.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Therefore BCYX.
By 12, BCYX . BCXA . ⊃ . CYXA. By G, ABCY . ⊃ . CYAB.
By 12, CYXA . CYAB . ⊃ . YXAB, whence by G, ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 7. *Proof of* 14 *from* G, 12, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . CYAB.
By 19, ABCX . ABCY . ⊃ . ACXY ∨ BCYX.

Case 1. If ACXY, then by G, CXYA. By G, ABCX . ⊃ . CXAB.
By 12, CXYA . CXAB . ⊃ . XYAB, whence by G, ABXY.

Case 2. If BCYX, then by 12, BCYX . BCXA . ⊃ . CYXA.
By 12, CYXA . CYAB . ⊃ . YXAB, whence by G, ABYX.
Therefore ABXY ∨ ABYX.

THEOREM 14. 8. *Proof of* 14 *from* G, 13, 10.

THEOREM 14. 9. *Proof of* 14 *from* G, 13, 15.

THEOREM 14. 10. *Proof of* 14 *from* G, 13, 16.

THEOREM 14. 11. *Proof of* 14 *from* G, 13, 17.

THEOREM 14. 12. *Proof of* 14 *from* G, 13, 18.

THEOREM 14. 13. *Proof of* 14 *from* G, 13, 19.

The six Theorems, 14. 8 to 14. 13, follow from Theorems 14. 2 to 14. 7, by the aid of Theorem 12. 2 (which says that Postulate 12 is a consequence of Postulates G and 13).

THEOREM 14. 14. *Proof of* 14 *from* G, 11, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
By G, ABCY . ⊃ . YABC. By G, ABCX . ⊃ . XABC.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. By G, YACX ∨ XACY.
If YACX, then by 11, YABC . YACX . ⊃ . YABX; by G, ABXY.
If XACY, then by 11, XABC . XACY . ⊃ . XABY; by G, ABYX.
Therefore, ABXY ∨ ABYX.

THEOREM 14. 15. *Proof of* 14 *from* G, 11, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
Suppose ABXY is false.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Therefore ACYX.
By G, ACYX . ⊃ . XACY. By G, ABCX . ⊃ . XABC.

By 11, xabc . xacy . ⊃ . xaby; by G, abyx.
Therefore abxy ∨ abyx.

Theorem 14. 16. *Proof of* 14 *from* G, H, 16.

To prove: abcx . abcy . ⊃ . abxy ∨ abyx.
Suppose abxy = 0 and abyx = 0. Then by G, xyab = 0 and xaby = 0.
By G, abcx . ⊃ . bcxa. By G, abcy . ⊃ . cyab.
By G, abcy . ⊃ . bcya. By G, abcx . ⊃ . cxab.
By 16, abcx . abcy . ⊃ . bcxy ∨ bcyx.

Case 1. Suppose bcxy. By G, bcxy . ⊃ . xybc.
By 16, bcxy . bcxa . ⊃ . cxya ∨ cxay. By G, bcxy . ⊃ . cxyb.
But if cxya, then by 16, cxya . cxyb . ⊃ . xyab ∨ xyba, whence xyba (since xyab = 0). Then by 16, xyba . xybc . ⊃ . ybac ∨ ybca, whence by G, cyba ∨ bcay, which conflicts with cyab and bcya by H.
Therefore cxay. By 16, cxay . cxab . ⊃ . xayb ∨ xaby, whence xayb (since xaby = 0). By G, cxay . ⊃ . xayc.
By 16, xayb . xayc . ⊃ . aybc ∨ aycb, whence by G, bcay ∨ aycb.
But bcay . vs . bcya, by H. Therefore aycb. By G, cxay . ⊃ . aycx.
By 16, aycb . aycx . ⊃ . ycbx ∨ ycxb. By G, xycb ∨ cxby.
But xycb . vs . xybc, by H; and cxby . vs . cxyb, by H.
Therefore Case 1 is impossible.

Case 2. Suppose bcyx. By similar reasoning (merely interchanging x and y), Case 2 is impossible.
Therefore abxy ∨ abyx.

Theorem 14. 17. *Proof of* 14 *from* H, 17.

To prove: abcx . abcy . ⊃ . abxy ∨ abyx.
Suppose both abxy and abyx are false.
By 17, abcx . abcy . ⊃ . abxy ∨ acyx. Hence acyx.
By 17, abcy . abcx . ⊃ . abyx ∨ acxy. Hence acxy.
But acyx . vs . acxy by H. Therefore abxy ∨ abyx.

Theorem 14. 18. *Proof of* 14 *from* H, 18.

To prove: abcx . abcy . ⊃ . abxy ∨ abyx.
Suppose both abxy and abyx are false.
By 18, abcx . abcy . ⊃ . abxy ∨ bcyx. Hence bcyx.
By 18, abcy . abcx . ⊃ . abyx ∨ bcxy. Hence bcxy.
But bcyx . vs . bcxy by H. Therefore abxy ∨ abyx.

THEOREM 14. 19. *Proof of* 14 *from* G, H, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY.

Case 1. If ACXY is true, then ACYX is false, by H. Therefore BCXY, whence by G, CXYB. By G, ACXY . ⊃ . CXYA.
By G, ABCY . ⊃ . CYAB.
By 19, CXYB . CXYA . ⊃ . CYBA ∪ XYAB.
But CYBA . vs . CYAB, by H. Therefore XYAB, whence by G, ABXY.

Case 2. If BCYX is true, then BCXY is false, by H. Therefore ACYX, whence by G, CYXA. By G, BCYX . ⊃ . CYXB.
By G, ABCX . ⊃ . CXAB.
By 19, CYXB . CYXA . ⊃ . CXBA ∪ YXAB.
But CXBA . vs . CXAB by H. Therefore YXAB, whence by G, ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 20. *Proof of* 14 *from* R, G, 16.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By R, ABCX . ⊃ . XCBA, whence by G, CBAX.
By R, ABCY . ⊃ . YCBA, whence by G, CBAY.
By 16, CBAX . CBAY . ⊃ . BAXY ∪ BAYX.
If BAXY, then by R, G, ABYX; and if BAYX, then by R, G, ABXY.
Therefore ABXY ∪ ABYX.

THEOREM 14. 21. *Proof of* 14 *from* R, G, 11, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By R, G, ABCX . ⊃ . CBAX. By R, G, ABCY . ⊃ . CBAY.
By G, ABCX . ⊃ . XABC.
By 19, CBAX . CBAY . ⊃ . CAXY ∪ BAYX, whence by R, G, XACY ∪ ABXY.
But if XACY, then by 11, XABC . XACY . ⊃ . XABY; by G, ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 22. *Proof of* 14 *from* R, G, 10, 15, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABXY = 0 and ABYX = 0. By G, BXYA = 0 and BYXA = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX. By G, XACY.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. Hence ACXY. By G, YACX.
By G, ABCY . ⊃ . BCYA.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX. Hence BCYX. By G, XBCY.

By 15, BCYX . BCYA . ⊃ . BYXA ∨ BYAX. Hence BYAX, whence by G, AXBY.
By G, ABCX . ⊃ . BCXA.
By 10. BCXA . Y . ⊃ . BYXA ∨ BCXY. Therefore BCXY, whence by G, YBCX.
By 15, BCXY . BCXA . ⊃ . BXYA ∨ BXAY. Therefore BXAY, whence by G, AYBX.
By 10, AXBY . C . ⊃ . ACBY ∨ AXBC.

Case 1. Suppose ACBY. Then by G, YACB, and by R, YBCA.
Since ABXY = 0 and ABYX = 0, by R, G, YABX = 0 and YBAX = 0.
By 17, YACB . YACX . ⊃ . YABX ∨ YCXB. Hence YCXB.
By 17, YBCA . YBCX . ⊃ . YBAX ∨ YCXA. Hence YCXA.
Then by 15, YCXB . YCXA . ⊃ . YXBA ∨ YXAB,
whence by R, G, ABXY ∨ ABYX.

Case 2. Suppose AXBC. Then by R, G, XACB, and by G, XBCA.
Since ABYX = 0 and ABXY = 0, by R, G, XABY = 0 and XBAY = 0.
By 17, XACB . XACY . ⊃ . XABY ∨ XCYB. Therefore XCYB.
By 17, XBCA . XBCY . ⊃ . XBAY ∨ XCYA. Therefore XCYA.
Then by 15, XCYA . XCYB . ⊃ . XYAB ∨ XYBA,
whence by G, R, ABXY ∨ ABYX.
Therefore ABXY ∨ ABYX.

THEOREM 14. 23. *Proof of* 14 *from* R, G, 10, 15, 19,

Postulate 17 follows from R, G, 19, as we shall see in Theorem 17. 27 (below). Postulate 14 follows from R, G, 10, 15, 17, by Theorem 14. 22.
Therefore 14 follows from R, G, 10, 15, 19.

THEOREM 14. 24. *Proof of* 14 *from* R, G, 15, 17, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
Suppose ABXY = 0 and ABYX = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Hence ACYX. By G, CYXA.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY. Hence BCXY.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Hence BCYX. By G, CYXB.
Since ABXY = 0 and ABYX = 0, by R, G, YXAB = 0 and AYXB = 0.
By 18, CYXB . CYXA . ⊃ . CYBA ∨ YXAB. Hence CYBA. By G, ACYB.
By 17, ACYB . ACYX . ⊃ . ACBX ∨ AYXB. Hence ACBX.
By G, ABCX . ⊃ . BCXA. Since ABXY = 0, by G, BXYA = 0.
By 17, BCXA . BCXY . ⊃ . BCAY ∨ BXYA. Hence BCAY. By R, G, ACBY.

By 15, ACBX . ACBY . ⊃ . ABXY ∪ ABYX.
Therefore ABXY ∪ ABYX.

THEOREM 14. 25. *Proof of* 14 *from* R, G, 15, 18, 19.

Postulate 17 follows from R, G, 19, as we shall see in Theorem 17. 27 (below). Postulate 14 follows from R, G, 15, 17, 18, by Theorem 14. 24. Therefore 14 follows from R, G, 15, 18, 19.

THEOREM 14. 26. *Proof of* 14 *from* R, F, G, 12.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By F, at least one of the twenty-four permutations of the elements A, B, X, Y will form a true tetrad. These twenty-four permutations may be arranged in three lines, as follows:

ABXY, BXYA, XYAB, YABX; XBAY, BAYX, AYXB, YXBA.
ABYX, BYXA, YXAB, XABY; YBAX, BAXY, AXYB, XYBA.
AXBY, XBYA, BYAX, YAXB; BXAY, XAYB, AYBX, YBXA.

By R, G, if any tetrad in the first line is true, then ABXY is true.
By R, G, if any tetrad in the second line is true, then ABYX is true.
By R, G, if any tetrad in the third line is true, then AXBY is true.
Hence, by F, R, G, we have ABXY ∪ ABYX ∪ AXBY.
Suppose both ABXY and ABYX are false. Then AXBY.
By R, G, ABCX . ⊃ . AXCB. By R, G, ABCY . ⊃ . CBAY.
By 12, AXCB . AXBY . ⊃ . XCBY, whence by G, CBYX.
By 12, CBAY . CBYX . ⊃ . BAYX; By R, G, ABYX; which is false.
Therefore ABXY ∪ ABYX.

THEOREM 14. 27. *Proof of* 14 *from* R, F, G, 13.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By F, R, G (as in the preceding theorem), ABXY ∪ ABYX ∪ AXBY.
Suppose both ABXY and ABYX are false.
Then AXBY, whence by G, YABX. By G, ABCY . ⊃ . YABC.
By 13, YAXB . YABC . ⊃ . YXBC, whence by G, BCYX.
By G, ABCX . ⊃ . BCXA.
By 13, BCYX . BCXA . ⊃ . BYXA, by R, G, ABYX; which is false.
Therefore ABXY ∪ ABYX.

THEOREM 14. 28. *Proof of* 14 *from* R, 10, 12, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
In view of Theorem 18. 30 (below), we may use Postulate 18 as well as R, 10, 12, 17 in this proof.

Suppose ABXY = 0 and ABYX = 0. By R, YXBA = 0 and XYBA = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Hence ACYX. By R, XYCA.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Hence BCYX. By R, XYCB.
By 17, XYCB . XYCA . ⊃ . XYBA ∨ XCAB. Hence XCAB.
By 18, XYCB . XYCA . ⊃ . XYBA ∨ YCAB. Hence YCAB.
By R, ABCX . ⊃ . XCBA. By R, ABCY . ⊃ . YCBA.
By 10, XCBA . Y . ⊃ . XYBA ∨ XCBY. Hence XCBY.
By 10, YCBA . X . ⊃ . YXBA ∨ YCBX. Hence YCBX.
By 12, XCAB . XCBY . ⊃ . CABY.
By 12, YCAB . YCBX . ⊃ . CABX.
By 18, CABX . CABY . ⊃ . CAXY ∨ ABYX. Hence CAXY.
By 12, CABX . CAXY . ⊃ . ABXY. But ABXY = 0.
Therefore ABXY ∨ ABYX.

THEOREM 14. 29. *Proof of* 14 *from* R, 10, 11, 13, 15, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
In view of Theorem 17. 37 (below), we may use Postulate 17 in this proof, as well as R, 10, 11, 13, 15, 18.
Suppose ABXY = 0 and ABYX = 0. By R, YXBA = 0 and XYBA = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Hence ACYX. By R, XYCA.
By 17, ABCY . ABCX . ⊃ . ABYX ∨ ACXY. Hence ACXY. By R, YXCA.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Hence BCYX. By R, XYCB.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY. Hence BCXY. By R, YXCB.
By R, ABCX . ⊃ . XCBA. By R, ABCY . ⊃ . YCBA.
By 10, XCBA . Y . ⊃ . XYBA ∨ XCBY. Hence XCBY. By R, YBCX.
By 10, YCBA . X . ⊃ . YXBA ∨ YCBX. Hence YCBX. By R, XBCY.
Now ACBX = 0 (for if not, by 13, ACBX . ACXY . ⊃ . ABXY). By R, XBCA = 0.
Also, ACBY = 0 (for if not, by 13, ACBY . ACYX . ⊃ . ABYX). By R, YBCA = 0.
By 10, XBCY . A . ⊃ . XACY ∨ XBCA. Hence XACY. By R, YCAX.
By 10, YBCX . A . ⊃ . YACX ∨ YBCA. Hence YACX. By R, XCAY.
Now XCYB = 0 (for if not, by 13, XCYB . XCBA . ⊃ . XYBA).
Also, YCXB = 0 (for if not, by 13, YCXB . YCBA . ⊃ . YXBA).
By 18, XYCB . XYCA . ⊃ . XYBA ∨ YCAB. Hence YCAB.
By 18, YXCB . YXCA . ⊃ . YXBA ∨ XCAB. Hence XCAB.
By 18, YCAX . YCAB . ⊃ . YCXB ∨ CABX. Hence CABX.
By 18, XCAY . XCAB . ⊃ . XCYB ∨ CABY. Hence CABY.
By 15, CABX . CABY . ⊃ . CBXY ∨ CBYX. By R, YXBC ∨ XYBC.
If YXBC, then by 11, YXBC . YXCA . ⊃ . YXBA, which is false.

If XYBC, then by 11, XYBC . XYCA . ⊃ . XYBA, which is false.
Therefore, ABXY ∪ ABYX.

THEOREM 14. 30. *Proof of* 14 *from* R, 10, 13, 17, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABXY = 0 and ABYX = 0. By R, YXBA = 0 and XYBA = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX. By R, XYCA.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. Hence ACXY.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence BCYX. By R, XYCB.
By 17, XYCB . XYCA . ⊃ . XYBA ∪ XCAB. Hence XCAB.
By 18, XYCB . XYCA . ⊃ . XYBA ∪ YCAB. Hence YCAB.
By R, ABCX . ⊃ . XCBA. By R, ABCY . ⊃ . YCBA.
By 10, XCBA . Y . ⊃ . XYBA ∪ XCBY. Hence XCBY. By R, YBCX.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. Hence YCBX. By R, XBCY.
Now ACBY = 0 (for if not, by 13, ACBY . ACYX . ⊃ . ABYX). By R, YBCA = 0.
Also, ACBX = 0 (for if not, by 13, ACBX . ACXY . ⊃ . ABXY). By R, XBCA = 0.
By 10, YBCX . A . ⊃ . YACX ∪ YBCA. Hence YACX. By R, XCAY.
By 10, XBCY . A . ⊃ . XACY ∪ XBCA. Hence XACY. By R, YCAX.
Now XCYB = 0 (for if not, by 13, XCYB . XCBA . ⊃ . XYBA).
And YCXB = 0 (for if not, by 13, YCXB . YCBA . ⊃ . YXBA).
By 18, XCAY . XCAB . ⊃ . XCYB ∪ CABY. Hence CABY.
By 18, YCAX . YCAB . ⊃ . YCXB ∪ CABX. Hence CABX.
By 18, CABY . CABX . ⊃ . CAYX ∪ ABXY. Hence CAYX.
By 13, CABY . CAYX . ⊃ . CBYX. By R, XYBC.
By 13, XYBC . XYCA . ⊃ . XBCA, which is false.
Therefore ABXY ∪ ABYX.

THEOREM 14. 31. *Proof of* 14 *from* R, 10, 12, 15, 18.

The proof of this theorem is left to the reader.

THEOREM 14. 32. *Proof of* 14 *from* R, 10, 13, 17, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
Suppose ABXY = 0 and ABYX = 0.
By R, XCBA, YCBA, YXBA = 0, XYBA = 0.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. But YXBA = 0. Hence YCBX. By R, XBCY.
By 10, XCBA . Y . ⊃ . XYBA ∪ XCBY. But XYBA = 0. Hence XCBY. By R, YBCX.

By 13, if XCYB, then XCYB . XCBA . ⊃ . XYBA, which = 0. Hence XCYB = 0.
By 13, if YCXB, then YCXB . YCBA . ⊃ . YXBA, which = 0. Hence YCXB = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. But ABXY = 0. Hence ACYX. By R, XYCA.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. But ABYX = 0. Hence ACXY. By R, YXCA.
By 13 and R, if YBCA, then ACBY . ACYX . ⊃ . ABYX, which = 0. Hence YBCA = 0.
By 13 and R, if XBCA, then ACBX . ACXY . ⊃ . ABXY, which = 0. Hence XBCA = 0.
By 10, XBCY . A . ⊃ . XACY ∪ XBCA. But XBCA = 0. Hence XACY. By R, YCAX.
By 10, XYCA . B . ⊃ . XBCA ∪ XYCB. But XBCA = 0. Hence XYCB.
By 10, YBCX . A . ⊃ . YACX ∪ YBCA. But YBCA = 0. Hence YACX. By R, XCAY.
By 10, YXCA . B . ⊃ . YBCA ∪ YXCB. But YBCA = 0. Hence YXCB. By R, BCXY.
By 17, YXCB . YXCA . ⊃ . YXBA ∪ YCAB. But YXBA = 0. Hence YCAB.
By 17, XYCB . XYCA . ⊃ . XYBA ∪ XCAB. But XYBA = 0. Hence XCAB.
By 13, if XAYB, then XACY . XAYB . ⊃ . XCYB, which = 0. Hence XAYB = 0.
By 13, if YAXB, then YACX . YAXB . ⊃ . YCXB, which = 0. Hence YAXB = 0.
By 13 and R, if CBXY, then YXBC . YXCA . ⊃ . YBCA, which = 0. Hence CBXY = 0.
By 19, XCAY . XCAB . ⊃ . XAYB ∪ CABY. But XAYB = 0. Hence CABY.
By 19, YCAX . YCAB . ⊃ . YAXB ∪ CABX. But YAXB = 0. Hence CABX.
By 19, CABX . CABY . ⊃ . CBXY ∪ ABYX. But CBXY = 0. And ABYX = 0.
Hence the supposition is false. Therefore ABXY ∪ ABYX.

Forty-Five Proofs of Postulate 15.

Theorem 15. 1. *Proof of* 15 *from* G, H, 10.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCX . ⊃ . XABC. By G, ABCY . ⊃ . BCYA.
By 10, XABC . Y . ⊃ . XYBC ∪ XABY, whence by G, YBCX ∪ ABYX.

Case 1. If YBCX, then by 10, YBCX . A . ⊃ . YACX ∪ YBCA, whence by G. ACXY ∪ BCAY. But BCYA . VS . BCAY, by H. Therefore ACXY.

Case 2. If ABYX, then by 10, ABYX . C . ⊃ . ACYX ∪ ABYC.
But ABYC . vs . ABCY, by H. Therefore ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 2. *Proof of* 15 *from* G, 10, 12.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . BCXA.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX.

Case 1. If BXYA, then by G, ABXY. By 12, ABCX . ABXY . ⊃ . BCXY.
By 12, BCXY . BCYA . ⊃ . CXYA, whence by G, ACXY.

Case 2. If BCYX, then by G, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 3. *Proof of* 15 *from* G, 12, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . BCXA.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.

Case 1. If ABXY, then by 12, ABCX . ABXY . ⊃ . BCXY.
By 12, BCXY . BCYA . ⊃ . CXYA, whence by G, ACXY.

Case 2. If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX.
By 12, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 4. *Proof of* 15 *from* G, 12, 16.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . BCXA.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
If BCXY, then by 12, BCXY . BCYA . ⊃ . CXYA, whence by G, ACXY.
If BCYX, then by 12, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 5. *Proof of* 15 *from* G, 12, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY.
If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX. By G, ABCX . ⊃ . BCXA.
By 12, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 6. *Proof of* 15 *from* G, 12, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCY . ⊃ . BCYA. By G, ABCX ⊃ . BCXA.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX.

Case 1. If ABXY, then by 12, ABCX . ABXY . ⊃ . BCXY.
By 12, BCXY . BCYA . ⊃ . CXYA, whence by G, ACXY.

Case 2. If BCYX, then by 12, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 7. *Proof of* 15 *from* G, 12, 19.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
Suppose ACXY is false. By G, ABCX . ⊃ . BCXA.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Therefore BCYX.
By 12, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 8. *Proof of* 15 *from* G, 10, 13.

This follows from Theorem 15. 2, by aid of Theorem 12. 2.

THEOREM 15. 9. *Proof of* 15 *from* 13, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
If ABXY, then by 13, ABCX . ABXY . ⊃ . ACXY.
If ABYX, then by 13, ABCY . ABYX . ⊃ . ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 10. *Proof of* 15 *from* G, 13, 16.

This follows from Theorem 15. 4, by aid of Theorem 12. 2.

THEOREM 15. 11. *Proof of* 15 *from* 13, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
But if ABXY, then by 13, ABCX . ABXY . ⊃ . ACXY.
Therefore ACXY ∪ ACYX.

THEOREM 15. 12. *Proof of* 15 *from* G, 13, 18.

THEOREM 15. 13. *Proof of* 15 *from* G, 13, 19.

These theorems follow from Theorems 15. 6 and 15. 7 by aid of Theorem 12. 2.

THEOREM 15. 14. *Proof of* 15 *from* H, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
Suppose both ACXY and ACYX are false.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Therefore ABXY.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. Therefore ABYX.
But ABXY . vs . ABYX, by H. Therefore ACXY ∪ ACYX.

THEOREM 15. 15. *Proof of* 15 *from* G, H, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX.

Case 1. Suppose ABXY. Then by G, YABX. By G, ABCY . ⊃ . YABC.
By 18, YABC . YABX . ⊃ . YACX ∪ ABXC, whence by G, ACXY ∪ ABXC.
But ABXC . vs . ABCX, by H. Therefore ACXY.

Case 2. Suppose BCYX. By G, ABCY . ⊃ . BCYA.
By G, ABCX . ⊃ . BCXA.
By 18, BCYA . BCYX . ⊃ . BCAX ∪ CYXA, whence by G, BCAX ∪ ACYX.
But BCAX . vs . BCXA, by H. Therefore ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 16. *Proof of* 15 *from* H, 19.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
Suppose both ACXY and ACYX are false.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Therefore BCYX.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY. Therefore BCXY.
But BCYX . vs . BCXY, by H. Therefore ACXY ∪ ACYX.

THEOREM 15. 17. *Proof of* 15 *from* G, 10, 11, 14, 16.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . BCYA.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
By 10, BCXA . Y . ⊃ . BYXA ∪ BCXY, whence by G, ABYX ∪ BCXY.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX, whence by G, ABXY ∪ BCYX.

Case 1. Suppose both ABXY and BCXY are true. Then by G, XYAB and XYBC.
By 11, XYAB . XYBC . ⊃ . XYAC, whence by G, ACXY.

Case 2. Suppose either ABXY or BCXY is false. Then ABYX and BCYX are true, whence by G, YXAB and YXBC are true.
By 11, YXAB . YXBC . ⊃ . YXAC, whence by G, ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 18. *Proof of* 15 *from* G, 11, 14, 16, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose ACXY = 0 and ACYX = 0.
Then by G, XYAC = 0 and YXAC = 0.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX, whence by G, XYAB ∨ YXAB.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX, whence by G, XYAB ∨ YXBC.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY, whence by G, YXAB ∨ XYBC.
By 16, ABCY . ABCX . ⊃ . BCYX ∨ BCXY, whence by G, YXBC ∨ XYBC.

Case 1. Suppose XYAB is false. Then YXAB and YXBC are true.
Then by 11, YXAB . YXBC . ⊃ . YXAC, which is false.

Case 2. Suppose XYBC is false. Then YXAB and YXBC are true.
Then by 11, YXAB . YXBC . ⊃ . YXAC, which is false.
Therefore XYAB and XYBC are true.
By 11, XYAB . XYBC . ⊃ . XYAC, whence by G, ACXY, which is false.
Therefore ACXY ∨ ACYX.

THEOREM 15. 19. *Proof of* 15 *from* G, 11, 14, 19.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose both ACXY and ACYX are false.
By 19, ABCX . ABCY . ⊃ . ACXY ∨ BCYX. Then BCYX; by G, YXBC.
By 19, ABCY . ABCX . ⊃ . ACYX ∨ BCXY. Then BCXY; by G, XYBC.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX, whence by G, XYAB ∨ YXAB.
If XYAB, then by 11, XYAB . XYBC . ⊃ . XYAC; by G, ACXY.
If YXAB, then by 11, YXAB . YXBC . ⊃ . YXAC; by G, ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 20. *Proof of* 15 *from* G, 11, 16, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose both ACXY and ACYX are false.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Then ABXY; by G, XYAB.

By 17, ABCY . ABCX . ⊃ . ABYX ∨ ACXY. Then ABYX; by G, YXAB.
By 16, ABCX . ABCY . ⊃ . BCXY ∨ BCYX; by G, XYBC ∨ YXBC.
If XYBC, then by 11, XYAB . XYBC . ⊃ . XYAC; by G, ACXY.
If YXBC, then by 11, YXAB . YXBC . ⊃ . YXAC; by G, ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 21. *Proof of* 15 *from* G, 11, 17, 19.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose ACYX is false.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Then ABXY; by G, XYAB.
By 19, ABCY . ABCX . ⊃ . ACYX ∨ BCXY. Then BCXY; by G, XYBC.
By 11, XYAB . XYBC . ⊃ . XYAC, whence by G, ACXY.
Therefore ACXY ∨ ACYX.

THEOREM 15. 22. *Proof of* 15 *from* H, 10, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX.

Case 1. If ABXY, then by 10, ABXY . C . ⊃ . ACXY ∨ ABXC.
But ABXC . VS . ABCX, by H. Therefore ACXY.

Case 2. If ABYX, then by 10, ABYX . C . ⊃ . ACYX ∨ ABYC.
But ABYC . VS . ABCY, by H. Therefore ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 23. *Proof of* 15 *from* H, 10, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY.

Case 1. If ABXY, then by 10, ABXY . C . ⊃ . ACXY ∨ ABXC.
But ABXC . VS . ABCX, by H. Therefore ACXY.

Case 2. If BCYX, then by H, BCXY is false. Then ABYX is true.
Then by 10, ABYX . C . ⊃ . ACYX ∨ ABYC.
But ABYC . VS . ABCY, by H. Therefore ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 24. *Proof of* 15 *from* H, 13, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY.

Case 1. If ABXY, then by 13, ABCX . ABXY . ⊃ . ACXY.

Case 2. If BCYX, then by H, BCXY is false. Then ABYX is true.
Then by 13, ABCY . ABYX . ⊃ . ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 25. *Proof of* 15 *from* R, G, 11, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
By R, G, ABCX . ⊃ . CBAX. By R, G, ABCY . ⊃ . CBAY.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
By 14, CBAX . CBAY . ⊃ . CBXY ∨ CBYX.

Case 1. If ABXY and CBYX, by R, G, XYAB and XYBC.
By 11, XYAB . XYBC . ⊃ . XYAC. By G, ACXY.

Case 2. If ABYX and CBXY, by R, G, YXAB and YXBC.
By 11, YXAB . YXBC . ⊃ . YXAC. By G, ACYX.

Case 3. If ABXY and CBXY, by R, YXBA and YXBC.
By 14, YXBA. YXBC . ⊃ . YXAC ∨ YXCA . By R, G, ACYX ∨ ACXY.
Case 4. If ABYX and CBYX, by R, XYBA and XYBC.
By 14, XYBA . XYBC . ⊃ . XYAC ∨ XYCA. By R, G, ACXY ∨ ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 26. *Proof of* 15 *from* R, G, 11, 16.

Proof: 14 from R, G, 16, by Theorem 14. 20; then 15 from R, G, 11, 14 by Theorem 15. 25.

THEOREM 15. 27. *Proof of* 15 *from* R, G, 11, 17.

Proof: 14 from G, 11, 17 by Theorem 14. 15; then 15 from R, G, 11, 14 by Theorem 15. 25.

THEOREM 15. 28. *Proof of* 15 *from* R, G, 11, 19.

Proof: 14 from R, G, 11, 19 by Theorem 14. 21; then 15 from R, G, 11, 14 by Theorem 15. 25.

THEOREM 15. 29. *Proof of* 15 *from* R, G, 14, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose ACXY = 0 and ACYX = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Hence ABXY. By R, YXBA.
By R, G, ABCX . ⊃ . CBAX. By R, G, ABCY . ⊃ . CBAY.

By 17, CBAX . CBAY . ⊃ . CBXY ∪ CAYX. By R, G, YXBC ∪ ACXY. Hence YXBC.
By 14, YXBA . YXBC . ⊃ . YXAC ∪ YXCA. By R, G, ACYX ∪ ACXY.
Therefore ACXY ∪ ACYX.

THEOREM 15. 30. *Proof of* 15 *from* R, G, 14, 19.

Proof: 17 from R, G, 19 by Theorem 17. 27 (below); then 15 from R, G, 14, 17 by Theorem 15. 29.

THEOREM 15. 31. *Proof of* 15 *from* R, G, 16, 17.

Proof: 14 from R, G, 16 by Theorem 14. 20; then 15 from R, G, 14, 17 by Theorem 15. 29.

THEOREM 15. 32. *Proof of* 15 *from* R, G, 16, 19.

Proof: 17 from R, G, 19 by Theorem 17. 27 (below); then 15 from R, G, 16, 17 by Theorem 15. 31.

THEOREM 15. 33. *Proof of* 15 *from* R, G, H, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
By G, ABCY . ⊃ . YABC. By G, ABCX . ⊃ . XABC.
By R, G, ABCX . ⊃ . AXCB. By R, G, ABCY . ⊃ . AYCB.
Suppose ACXY = o and ACYX = o.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.

Case 1. If ABXY, then by G, YABX.
By 14, YABX . YABC . ⊃ . YAXC ∪ YACX. By G, AXCY ∪ ACXY. Hence AXCY.
By 14, AXCY . AXCB . ⊃ . AXYB ∪ AXBY. By R, G, ABYX ∪ YAXB.
But ABYX . vs . ABXY by H; and YAXB . vs . YABX, by H.

Case 2. If ABYX, then by G, XABY.
By 14, XABY . XABC . ⊃ . XAYC ∪ XACY. By G, AYCX ∪ ACYX. Hence AYCX.
By 14, AYCX . AYCB . ⊃ . AYXB ∪ AYBX. By R, G, ABXY ∪ XAYB.
But ABXY . vs . ABYX, by H; and XAYB . vs . XABY, by H.
Therefore ACXY ∪ ACYX.

THEOREM 15. 34. *Proof of* 15 *from* R, G, H, 16.

Proof: 14 from R, G, 16 by Theorem 14. 20; then 15 from R, G, H, 14. by Theorem 15. 33.

THEOREM 15. 35. *Proof of* 15 *from* R, F, G, 12.

Proof: 14 from R, F, G, 12 by Theorem 14. 26; then 15 from G, 12, 14 by Theorem 15. 3.

THEOREM 15. 36. *Proof of* 15 *from* R, F, G, 13.

Proof: 12 from G, 13 by Theorem 12. 2; hence 15 from R, F, G, 12, by Theorem 15. 35.

THEOREM 15. 37. *Proof of* 15 *from* R, H, 11, 18.

To prove: ABCX . ABCY ⊃ . ACXY ∨ ACYX.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY.

Case 1. If ABXY is true, then ABYX = o, by H. Hence BCXY.
By R, ABXY . ⊃ . YXBA. By R, BCXY . ⊃ . YXCB.
By 11, YXCB . YXBA . ⊃ . YXCA. By R, ACXY.

Case 2. If BCYX is true, then BCXY = o, by H. Hence ABYX.
By R, BCYX . ⊃ . XYCB. By R, ABYX . ⊃ . XYBA.
By 11, XYCB . XYBA . ⊃ . XYCA. By R, ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 38. *Proof of* 15 *from* R, 11, 12, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX.

Case 1. If ABXY, then by 12, ABCX . ABXY . ⊃ . BCXY. By R, YXCB.
By R, ABXY . ⊃ . YXBA.
By 11, YXCB . YXBA . ⊃ . YXCA. By R, ACXY.

Case 2. If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX. By R, XYCB.
By R, ABYX . ⊃ . XYBA.
By 11, XYCB . XYBA . ⊃ . XYCA. By R, ACYX.
Therefore ACXY ∨ ACYX.

THEOREM 15. 39. *Proof of* 15 *from* R, 11, 12, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose ACYX = o.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Hence ABXY. By R, YXBA.
By 12, ABCX . ABXY . ⊃ . BCXY. By R, YXCB.
By 11, YXCB . YXBA . ⊃ . YXCA. By R, ACXY.
Therefore ACXY ∨ ACYX.

THEOREM 15. 40. *Proof of* 15 *from* R, 11, 17, 19.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
Suppose ACYX = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ABXY. By R, YXBA.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY. Hence BCXY. By R, YXCB.
By 11, YXCB . XYBA . ⊃ . YXCA. By R, ACXY.
Therefore ACXY ∪ ACYX.

THEOREM 15. 41. *Proof of* 15 *from* R, 11, 14, 19.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
Suppose ACXY = 0 and ACYX = 0.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence BCYX. By R, XYCB.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY. Hence BCXY. By R, YXCB.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. By R, YXBA ∪ XYBA.
If YXBA, by 11, YXCB . YXBA . ⊃ . YXCA; by R, ACXY.
If XYBA, by 11, XYCB . XYBA . ⊃ . XYCA; by R, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 42. *Proof of* 15 *from* R, 11, 16, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ ACYX.
Suppose ACXY = 0 and ACYX = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ABXY. By R, YXBA.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. Hence ABYX. By R, XYBA.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. By R, YXCB ∪ XYCB.
If YXCB, by 11, YXCB . YXBA . ⊃ . YXCA. By R, ACXY.
If XYCB, by 11, XYCB . XYBA . ⊃ . XYCA. By R, ACYX.
Therefore ACXY ∪ ACYX.

THEOREM 15. 43. *Proof of* 15 *from* R, 10, 12, 17.

Proof: Postulate 11 follows from R, 10, 12, 17, by Theorem 11. 6; then 15 from R, 11, 12, 17, by Theorem 15. 39.

THEOREM 15. 44. *Proof of* 15 *from* R, H, 10, 12, 16.

Proof: Postulate 19 follows from R, H, 10, 12, 16, by Theorem 19. 37 (below); then 15 follows from H, 19, by Theorem 15. 16.

THEOREM 15. 45. *Proof of* 15 *from* R, 10, 12, 14.

The proof of this theorem is left to the reader.

Thirty-One Proofs of Postulate 16.

Theorem 16. 1. *Proof of* 16 *from* G, H, 10.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By G, ABCX . ⊃ . BCXA; and by G, ABCY . ⊃ . BCYA.
Suppose both BCXY and BCYX are false.
By 10, BCXA . Y . ⊃ . BYXA ∨ BCXY. Hence BYXA. By G, ABYX.
By 10, BCYA . X . ⊃ . BXYA ∨ BCYX. Hence BXYA. By G, ABXY.
But ABYX . VS . ABXY, by H.
Therefore BCXY ∨ BCYX.

Theorem 16. 2. *Proof of* 16 *from* G, 10, 12.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By G, ABCY . ⊃ . BCYA.
By 10, BCYA . X . ⊃ . BXYA ∨ BCYX.
If BXYA, then by G, ABXY. By 12, ABCX . ABXY . ⊃ . BCXY.
Therefore BCXY ∨ BCYX.

Theorem 16. 3. *Proof of* 16 *from* 12, 14.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
If ABXY, then by 12, ABCX . ABXY . ⊃ . BCXY.
If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX.
Therefore BCXY ∨ BCYX.

Theorem 16. 4. *Proof of* 16 *from* G, 12, 15.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By G, ABCX . ⊃ . CXAB. By G, ABCY . ⊃ . CYAB.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. By G, CXYA ∨ CYXA.

Case 1. If CXYA, by 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.
By 12, ABCX . ABXY . ⊃ . BCXY.

Case 2. If CYXA, by 12, CYXA . CYAB . ⊃ . YXAB. By G, ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore BCXY ∨ BCYX.

Theorem 16. 5. *Proof of* 16 *from* G, 12, 17.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX.

Case 1. If ABXY, by 12, ABCX . ABXY . ⊃ . BCXY.

Case 2. If ACYX, by G, CYXA. By G, ABCY . ⊃ . CYAB.
By 12, CYXA . CYAB . ⊃ . YXAB. By G, ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore BCXY ∪ BCYX.

THEOREM 16. 6. *Proof of* 16 *from* 12, 18.

To prove: ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
If ABXY, by 12, ABCX . ABXY . ⊃ . BCXY.
Therefore BCXY ∪ BCYX.

THEOREM 16. 7. *Proof of* 16 *from* G, 12, 19.

To prove: ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
Suppose BCYX is false.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY. By G, CXYA.
By G, ABCX . ⊃ . CXAB.
By 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.
By 12, ABCX . ABXY . ⊃ . BCXY.
Therefore BCYX ∪ BCXY.

THEOREM 16. 8. *Proof of* 16 *from* G, 10, 13.

THEOREM 16. 9. *Proof of* 16 *from* G, 13, 14.

THEOREM 16. 10. *Proof of* 16 *from* G, 13, 15.

THEOREM 16. 11. *Proof of* 16 *from* G, 13, 17.

THEOREM 16. 12. *Proof of* 16 *from* G, 13, 18.

THEOREM 16. 13. *Proof of* 16 *from* G, 13, 19.

These six theorems follow from Theorems 16. 2–16. 7, by the aid of Theorem 12. 2 (which says that 12 follows from G, 13).

THEOREM 16. 14. *Proof of* 16 *from* G, 11, 15.

To prove: ABCX . ABCY . BCXY ∪ BCYX.
By G, ABCX . ⊃ . CXAB. By G, ABCY . ⊃ . CYAB.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. By G, CXYA ∪ CYXA.

Case 1. If CXYA, then by 11, CXYA . CXAB . ⊃ . CXYB. By G, BCXY.

Case 2. If CYXA, then by 11, CYXA . CYAB . ⊃ . CYXB. By G, BCYX.
Therefore BCXY ∨ BCYX.

THEOREM 16. 15. *Proof of* 16 *from* G, 11, 19.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
Suppose BCYX is false. By G, ABCX . ⊃ . CXAB.
By 19, ABCX . ABCY . ⊃ . ACXY ∨ BCYX. Hence ACXY. By G, CXYA.
By 11, CXYA . CXAB . ⊃ . CXYB. By G, BCXY.
Therefore BCXY ∨ BCYX.

THEOREM 16. 16. *Proof of* 16 *from* G, H, 14.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By G, ABCY . ⊃ . YABC. By G, ABCX . ⊃ . XABC.
By G, ABCY . ⊃ . CYAB. By G, ABCY . ⊃ . BCYA.
Suppose BCXY = 0 and BCYX = 0. By G, CYXB = 0 and XYBC = 0.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX. By G, YABX ∨ XABY.
If YABX, then by 14, YABX . YABC . ⊃ . YAXC ∨ YACX.
If XABY, then by 14, XABY . XABC . ⊃ . XAYC ∨ XACY.

Case 1. Suppose YABX and YAXC.
By G, YAXC . ⊃ . CYAX. By G, YAXC . ⊃ . XCYA.
By 14, CYAX . CYAB . ⊃ . CYXB ∨ CYBX. Hence CYBX. By G, XCYB.
By 14, XCYB . XCYA . ⊃ . XCBA ∨ XCAB. By G, AXCB ∨ ABXC.
But ABXC . vs . ABCX, by H. Hence AXCB.
By G, YAXC . ⊃ . AXCY. By G, YABX . ⊃ . XYAB.
By 14, AXCY . AXCB . ⊃ . AXYB ∨ AXBY. By G, XYBA ∨ YAXB.
But XYBA . vs . XYAB, by H; and YAXB . vs . YABX, by H.
Therefore Case 1 is impossible.

Case 2. Suppose YABX and YACX.
By G, YABX . ⊃ . XYAB. By G, YACX . ⊃ . XYAC.
By G, YABX . ⊃ . BXYA.
By 14, XYAB . XYAC . ⊃ . XYBC ∨ XYCB. Hence XYCB. By G, BXYC.
By 14, BXYC . BXYA . ⊃ . BXCA ∨ BXAC. By G, ABXC ∨ XACB.
But ABXC . vs . ABCX, by H; and XACB . vs . XABC, by H.
Therefore Case 2 is impossible.

Case 3. Suppose XABY and XAYC.
By interchanging X and Y in Case 1, we see that Case 3 is impossible.

Case 4. Suppose XABY and XACY.
By interchanging X and Y in Case 2, we see that Case 4 is impossible.
Therefore BCXY ∨ BCYX.

THEOREM 16. 17. *Proof of* 16 *from* G, H, 17.

To prove: ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. By G, XYAB ∪ YXAC.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. By G, YXAB ∪ XYAC.

Case 1. Suppose XYAB and XYAC. By G, ABCX . ⊃ . XABC.
By 17, XYAB . XYAC . ⊃ . XYBC ∪ XACB.
But XACB . vs . XABC, by H. Hence XYBC. By G, BCXY.

Case 2. Suppose YXAB and YXAC. By G, ABCY . ⊃ . YABC.
By 17, YXAB . YXAC . ⊃ . YXBC ∪ YACB.
But YACB . vs . YABC, by H. Hence YXBC. By G, BCYX.

Case 3. Suppose XYAB and YXAB. By G, ABXY and ABYX.
But ABXY . vs . ABYX, by H.

Case 4. Suppose YXAC and XYAC. By G, ACYX and ACXY.
But ACYX . vs . ACXY, by H.
Therefore BCXY ∪ BCYX.

THEOREM 16. 18. *Proof of* 16 *from* H, 18.

To prove: ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
Suppose both BCXY and BCYX are false.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence ABXY.
By 18, ABCY . ABCX . ⊃ . ABYX ∪ BCXY. Hence ABYX.
But ABXY . vs . ABYX, by H.
Therefore BCXY ∪ BCYX.

THEOREM 16. 19. *Proof of* 16 *from* H, 19.

To prove: ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
Suppose both BCXY and BCYX are false.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY. Hence ACYX.
But ACXY . vs . ACYX, by H.
Therefore BCXY ∪ BCYX.

THEOREM 16. 20. *Proof of* 16 *from* H, 12, 17.

To prove: ABCX . ABCY . ⊃ . BCXY ∪ BCYX.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY.

Case 1. If ABXY, by 12, ABCX . ABXY . ⊃ . BCXY.

Case 2. If ABYX, by 12, ABCY . ABYX . ⊃ . BCYX.

Case 3. If both ABXY and ABYX are false, then ACYX and ACXY, which is impossible, by H.
Therefore BCXY ∨ BCYX.

THEOREM 16. 21. *Proof of* 16 *from* R, G, 14.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
By R, G, ABCX . ⊃ . CBAX. By R, G, ABCY . ⊃ . CBAY.
By 14, CBAX . CBAY . ⊃ . CBXY ∨ CBYX.
If CBXY, then by R, G, BCYX; and if CBYX, by R, G, BCXY.
Therefore BCXY ∨ BCYX.

THEOREM 16. 22. *Proof of* 16 *from* R, G, 11, 17.

Proof by Theorems 19. 26 and 16. 15.

THEOREM 16. 23. *Proof of* 16 *from* R, G, 10, 15, 17.

THEOREM 16. 24. *Proof of* 16 *from* R, G, 10, 15, 19.

THEOREM 16. 25. *Proof of* 16 *from* R, G, 15, 17, 18.

THEOREM 16. 26. *Proof of* 16 *from* R, G, 15, 18, 19.

THEOREM 16. 27. *Proof of* 16 *from* R, F, G, 12.

THEOREM 16. 28. *Proof of* 16 *from* R, F, G, 13.

These six theorems follow from Theorems 14. 21–14. 27 (which give postulate 14), by the aid of Theorem 16. 21.

THEOREM 16. 29. *Proof of* 16 *from* R, 10, 12, 19.

To prove: ABCX . ABCY . ⊃ . BCXY ∨ BCYX.
Suppose BCXY = 0 and BCYX = 0. By R, YXCB = 0 and XYCB = 0.
By 19, ABCX . ABCY . ⊃ . ACXY ∨ BCYX. Then ACXY. By R, YXCA.
By 19, ABCY . ABCX . ⊃ . ACYX ∨ BCXY. Then ACYX. By R, XYCA.
By 10, YXCA . B . ⊃ . YBCA ∨ YXCB. Then YBCA. By R, ACBY.
By 10, XYCA . B . ⊃ . XBCA ∨ XYCB. Then XBCA. By R, ACBX.
Now ABXY = 0 (for if not, by 12, ABCX . ABXY . ⊃ . BCXY). By R, YXBA = 0.

Also, ABYX = 0 (for if not, by 12, ABCY . ABYX . ⊃ . BCYX). By R, XYBA = 0.
By R, ABCX . ⊃ . XCBA.
By 10, XCBA . Y . ⊃ . XYBA ∪ XCBY. Then XCBY. By R, YBCX.
By R, ABCY . ⊃ . YCBA.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. Then YCBX. By R, XBCY.
By 19, ACBX . ACBY . ⊃ . ABXY ∪ CBYX. Then CBYX. By R, XYBC.
By 19, ACBY . ACBX . ⊃ . ABYX ∪ CBXY. Then CBXY. By R, YXBC.
By 10, XYBC . A . ⊃ . XABC ∪ XYBA. Then XABC. By R, CBAX.
By 10, YXBC . A . ⊃ . YABC ∪ YXBA. Then YABC. By R, CBAY.
By 10, ACXY . B . ⊃ . ABXY ∪ ACXB. Then ACXB.
By 10, ACYX . B . ⊃ . ABYX ∪ ACYB. Then ACYB.
By 12, ACXB . ACBY . ⊃ . CXBY.
By 12, ACYX . ACXB . ⊃ . CYXB.
By 12, ACXY . ACYB . ⊃ . CXYB.
Now CXBA = 0 (for if not, by 12, CXYB . CXBA . ⊃ . XYBA).
Also, CYBA = 0 (for if not, by 12, CYXB . CYBA . ⊃ . YXBA).
By 10, CXBY . A . ⊃ . CABY ∪ CXBA. Then CABY. By R, YBAC.
By 12, YBAC . YBCX . ⊃ . BACX.
By 12, CBAX . CBXY . ⊃ . BAXY.
By 12, CBAY . CBYX . ⊃ . BAYX.
By 10, BAXY . C . ⊃ . BCXY ∪ BAXC. Then BAXC.
By 10, BAYX . C . ⊃ . BCYX ∪ BAYC. Then BAYC.
By 12, BAYC . BACX . ⊃ . AYCX. By R, XCYA.
By 12, BAYX . BAXC . ⊃ . AYXC.
By R, ABCX . ⊃ . XCBA.
Now, XCYB = 0 (for if not, by 12, XCYB . XCBA . ⊃ . CYBA).
By 10, XCYA . B . ⊃ . XBYA ∪ XCYB. Then XBYA.
By 12, XBCY . XBYA . ⊃ . BCYA. By R, AYCB.
By 12, AYXC . AYCB . ⊃ . YXCB, which is false.
Therefore BCXY ∪ BCYX.

THEOREM 16. 30. *Proof of* 16 *from* R, 10, 12, 15.

Proof: 19 from R, 10, 12, 15, by Theorem 19. 38 (below) then 16 from R, 10, 12, 19, by Theorem 16. 29.

THEOREM 16. 31. *Proof of* 16 *from* R, 10, 12, 17.

Proof: 15 from R, 10, 12, 17, by Theorem 15. 43; then 16 from R, 10, 12, 15 by Theorem 16. 30.

THIRTY-NINE PROOFS OF POSTULATE 17.

THEOREM 17. 1. *Proof of* 17 *from* G, H, 10.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . BCYA.
Suppose ABXY is false. By G, BXYA is false.
By 10, BCYA . X . ⊃ . BXYA ∨ BCYX. Hence BCYX. By G, CYXB.
By 10, CYXB . A . ⊃ . CAXB ∨ CYXA. By G, BCAX ∨ ACYX.
But BCAX . vs . BCXA, by H. Hence ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 2. *Proof of* 17 *from* G, 10, 12.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . BCYA.
Suppose ABXY is false. Then by G, BXYA is false.
By 10, BCYA . X . ⊃ . BXYA ∨ BCYX. Hence BCYX.
By 12, BCYX . BCXA . ⊃ . CYXA. By G, ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 3. *Proof of* 17 *from* G, 12, 14.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX.
Suppose ABXY is false. Then ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX. By G, ABCX . ⊃ . BCXA.
By 12, BCYX . BCXA . ⊃ . CYXA. By G, ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 4. *Proof of* 17 *from* G, 12, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX.
Suppose ACYX is false. Hence ACXY. By G, CXYA.
By G, ABCX . ⊃ . CXAB.
By 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.
Therefore ABXY ∨ ACYX.

THEOREM 17. 5. *Proof of* 17 *from* G, 12, 16.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
By 16, ABCX . ABCY . ⊃ . BCXY ∨ BCYX. By G, ABCX . ⊃ . BCXA.
If BCYX is true, then by 12, BCYX . BCXA . ⊃ . CYXA, whence by G, ACYX.
By G, ABCY . ⊃ . BCYA.

If BCXY is true, then by 12, BCXY . BCYA . ⊃ . CXYA.
By G, ABCX . ⊃ . CXAB.
By 12, CXYA . CXAB . ⊃ . XYAB. By G, XYAB . ⊃ . ABXY.
Therefore ABXY ∪ ACYX.

THEOREM 17. 6. *Proof of* 17 *from* G, 12, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY is false. By G, ABCX . ⊃ . BCXA.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence BCYX.
By 12, BCYX . BCXA . ⊃ . CYXA. By G, ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 7. *Proof of* 17 *from* G, 12, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ACYX is false.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY. Hence BCXY.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . CXAB.
By 12, BCXY . BCYA . ⊃ . CXYA.
By 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.
Therefore ABXY ∪ ACYX.

THEOREM 17. 8. *Proof of* 17 *from* G, 10, 13.

This theorem follows from Theorem 17. 2 by aid of Theorem 12. 2.

THEOREM 17. 9. *Proof of* 17 *from* 13, 14.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
If ABYX, then by 13, ABCY . ABYX . ⊃ . ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 10. *Proof of* 17 *from* G, 13, 15.

THEOREM 17. 11. *Proof of* 17 *from* G, 13, 16.

THEOREM 17. 12. *Proof of* 17 *from* G, 13, 18.

THEOREM 17. 13. *Proof of* 17 *from* G, 13, 19.

These four theorems follow from Theorems 17. 4–17. 7 by aid of Theorem 12. 2.

THEOREM 17. 14. *Proof of* 17 *from* G, 11, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ACYX is false. By G, ABCY . ⊃ . YABC.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. Hence ACXY. By G, YACX.
By 11, YABC . YACX . ⊃ . YABX. By G, ABXY.
Therefore ABXY ∪ ACYX.

THEOREM 17. 15. *Proof of* 17 *from* G, 10, 11, 14.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
By G, ABCY . ⊃ . BCYA.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. By G, ABXY ∪ YXAB.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX. By G, ABXY ∪ YXBC.
Suppose ABXY is false. By 11, YXAB . YXBC . ⊃ . YXAC. By G, ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 16. *Proof of* 17 *from* G, 11, 14, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY is false.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX. By G, YXAB.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence BCYX. By G, YXBC.
By 11, YXAB . YXBC . ⊃ . YXAC. By G, ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 17. *Proof of* 17 *from* G, 11, 14, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY is false. By G, ABCY . ⊃ . YABC.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX. By G, YXAB.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. By G, YACX ∪ YXBC.

Case 1. If YACX, by 11, YABC . YACX . ⊃ . YABX. By G, ABXY, which is false.

Case 2. If YXBC, by 11, YXAB . YXBC . ⊃ . YXAC. By G, ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 18. *Proof of* 17 *from* G, H, 14, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY = 0 and ACYX = 0. By G, ABCX . ⊃ . XABC.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX. By G, XABY.

By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. Hence ACXY. By G, CXYA.
By 14, XABC . XABY . ⊃ . XACY ∨ XAYC. By G, ACYX ∨ CXAY.
But ACYX = 0; and CXAY . vs . CXYA, by H.
Therefore ABXY ∨ ACYX.

THEOREM 17. 19. *Proof of* 17 *from* G, H, 15, 16.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY = 0 and ACYX = 0. By G, XYAB = 0 and CYXA = 0.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. Hence ACXY. By G, CXYA.
By 16, ABCX . ABCY . ⊃ . BCXY ∨ BCYX.

Case 1. Suppose BCXY. By G, CXYB. By G, XYBC.
By G, ABCY . ⊃ . BCYA. By G, ABCY . ⊃ . CYAB.
By 16, CXYB . CXYA . ⊃ . XYBA ∨ XYAB. Hence XYBA.
By 16, XYBC . XYBA . ⊃ . YBCA ∨ YBAC . By G, BCAY ∨ CYBA.
But BCAY . vs . BCYA, by H; and CYBA . vs . CYAB, by H.

Case 2. Suppose BCYX. By G, ABCY . ⊃ . BCYA. By G, CYAB.
By G, ABCX . ⊃ . BCXA.
By 16, BCYX . BCYA . ⊃ . CYXA ∨ CYAX. Hence CYAX.
By 15, CYAX . CYAB . ⊃ . CAXB ∨ CABX. By G, BCAX ∨ ABXC.
But BCAX . vs . BCXA by H; and ABXC vs. ABCX, by H.
Therefore ABXY ∨ ACYX.

THEOREM 17. 20. *Proof of* 17 *from* G, H, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY is false. By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . BCXA.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Hence BCYX.
By 18, BCYA . BCYX . ⊃ . BCAX ∨ CYXA.
But BCAX . vs . BCXA. Hence CYXA. By G, ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 21. *Proof of* 17 *from* G, H, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY = 0 and ACYX = 0. By G, BXYA = 0.
By G, ABCX . ⊃ . BCXA. By G, CXAB. By G, ABCY . ⊃ . BCYA.
By 19, ABCY . ABCX . ⊃ . ACYX ∨ BCXY. Hence BCXY. By G, XYBC.
By 19, BCXY . BCXA . ⊃ . BXYA ∨ CXAY. Hence CXAY. By G, XAYC.
By 19, CXAB . CXAY . ⊃ . CABY ∨ XAYB. By G, ABYC ∨ XAYB.
But ABYC . vs . ABCY, by H. Hence XAYB.

By 19, XAYC . XAYB . ⊃ . XYCB ∪ AYBC. By G, XYCB ∪ BCAY.
But XYCB . vs . XYBC, by H; and BCAY . vs . BCYA, by H.
Therefore ABXY ∪ ACYX.

THEOREM 17. 22. *Proof of* 17 *from* H, 10, 14.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY is false.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX.
By 10, ABYX . C . ⊃ . ACYX ∪ ABYC.
But ABYC . vs . ABCY, by H. Hence ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 23. *Proof of* 17 *from* H, 10, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY is false.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence BCYX.
By 18, ABCY . ABCX . ⊃ . ABYX ∪ BCXY.
But BCXY . vs . BCYX, by H. Hence ABYX.
By 10, ABYX . C . ⊃ . ACYX ∪ ABYC.
But ABYC . vs . ABCY, by H. Hence ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 24. *Proof of* 17 *from* H, 18, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose both ABXY and ACYX are false.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence BCYX.
By 19, ABCY . ABCX . ⊃ . ACYX ∪ BCXY. Hence BCXY.
But BCXY . vs . BCYX by H.
Therefore ABXY ∪ ACYX.

THEOREM 17. 25. *Proof of* 17 *from* H, 13, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ ACYX.
Suppose ABXY is false.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence BCYX.
By 18, ABCY . ABCX . ⊃ . ABYX ∪ BCXY.
But BCXY . vs . BCYX, by H. Hence ABYX.
By 13, ABCY . ABYX . ⊃ . ACYX.
Therefore ABXY ∪ ACYX.

THEOREM 17. 26. *Proof of* 17 *from* H, 12, 14, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose both ABXY and ACYX are false.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX. Hence ABYX.
By 19, ABCY . ABCX . ⊃ . ACYX ∨ BCXY. Hence BCXY.
By 12, ABCY . ABYX . ⊃ . BCYX. But BCYX . vs . BCXY, by H.
Therefore ABXY ∨ ACYX.

THEOREM 17. 27. *Proof of* 17 *from* R, G, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
By R, G, ABCX . ⊃ . CBAX. By R, G, ABCY . ⊃ . CBAY.
By 19, CBAX . CBAY . ⊃ . CAXY ∨ BAYX.
But if CAXY, by R, G, ACYX; and if BAYX, by R, G, ABXY.
Therefore ABXY ∨ ACYX.

THEOREM 17. 28. *Proof of* 17 *from* R, G, 11, 14.

Proof: 15 from R, G, 11, 14, by Theorem 15. 25; then 17 from G, 11, 15 by Theorem 17. 4.

THEOREM 17. 29. *Proof of* 17 *from* R, G, 11, 16.

Proof: 15 from R, G, 11, 16 by Theorem 15. 16; then 17 from G, 11, 15 by Theorem 17. 14.

THEOREM 17. 30. *Proof of* 17 *from* R, G, H, 14.

Proof: 15 from R, G, H, 14 by Theorem 15. 33; then 17 from G, H, 14, 15 by Theorem 17. 18.

THEOREM 17. 31. *Proof of* 17 *from* R, G, H, 16.

Proof: 15 from R, G, H, 16 by Theorem 15. 34; then 17 from G, H, 15, 16 by Theorem 17. 19.

THEOREM 17. 32. *Proof of* 17 *from* R, F, G, 12.

Proof: 15 from R, F, G, 12 by Theorem 15. 35; then 17 from G, 12, 15 by Theorem 17. 4.

THEOREM 17. 33. *Proof of* 17 *from* R, F, G, 13.

Proof: 12 from G, 13 by Theorem 12. 2; then 17 from R, F, G, 12 by Theorem 17. 32.

THEOREM 17. 34. *Proof of* 17 *from* R, H, 11, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY = 0.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Hence BCYX. By R, XYCB.
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY. But BCXY . vs . BCYX, by H.
Hence ABYX. By R, XYBA.
By 11, XYCB . XYBA . ⊃ . XYCA. By R, ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 35. *Proof of* 17 *from* R, 11, 12, 14.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY = 0.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX. Then ABYX. By R, XYBA.
By 12, ABCY . ABYX . ⊃ . BCYX. By R, XYCB.
By 11, XYCB . XYBA . ⊃ . XYCA. By R, ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 36. *Proof of* 17 *from* R, 11, 14, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY = 0.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX. Then ABYX. By R, XYBA.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Then BCYX. By R, XYCB.
By 11, XYCB . XYBA . ⊃ . XYCA. By R, ACYX.
Therefore ABXY ∨ ACYX.

THEOREM 17. 37. *Proof of* 17 *from* R, 10, 11, 13, 15, 18.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ ACYX.
Suppose ABXY = 0 and ACYX = 0. By R, YXBA = 0 and XYCA = 0.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. Then ACXY. By R, YXCA.
By 18, ABCX . ABCY . ⊃ . ABXY ∨ BCYX. Then BCYX. By R, XYCB.
Now ABYX = 0 (for if not, by 13, ABCY . ABYX . ⊃ . ACYX).
By 18, ABCY . ABCX . ⊃ . ABYX ∨ BCXY. Then BCXY. By R, YXCB.
By 18, YXCB . YXCA . ⊃ . YXBA ∨ XCAB. Then XCAB. By R, BACX.
By 10, ACXY . B . ⊃ . ABXY ∨ ACXB. Then ACXB. By R, BXCA.
By 10, XYCB . A . ⊃ . XACB ∨ XYCA. Then XACB. By R, BCAX.
By R, ABCY . ⊃ . YCBA.
By 10, YCBA . X . ⊃ . YXBA ∨ YCBX. Then YCBX.
By 11, BCAX . BCXY . ⊃ . BCAY. By R, YACB.
By 13, BCAX . BCXY . ⊃ . BAXY.

By 11, BACX . BAXY . ⊃ . BACY. By R, YCAB.
By 13, YCAB . YCBX . ⊃ . YABX.
By 11, YACB . YABX . ⊃ . YACX.
Now YCXB = 0 (for if not, by 13, YCXB . YCBA . ⊃ . YXBA). By R, BXCY = 0.
By 10, BXCA . Y . ⊃ . BYCA ∪ BXCY. Then BYCA. By R, ACYB.
By 18, YACX . YACB . ⊃ . YAXB ∪ ACBX.
But if YAXB, then by R, BXAY, and by 11, BXCA . BXAY . ⊃ . BXCY, which is false, and if ACBX, then by 11, ACYB . ACBX . ⊃ . ACYX, which is false.
Therefore ABXY ∪ ACYX.

THEOREM 17. 38. *Proof of* 17 *from* R, 10, 11, 12, 15, 18.

Proof: 13 from R, 11, 12, by Theorem 13. 12; then 17 from R, 10, 11, 13, 15, 18, by Theorem 17. 37.

THEOREM 17. 39. *Proof of* 17 *from* R, 10, 12, 13, 15, 18.

Proof: 11 from R, 12, 13, by Theorem 11. 4; then 17 from R, 10, 11, 13, 15, 18, by Theorem 17. 37.

THIRTY-ONE PROOFS OF POSTULATE 18.

THEOREM 18. 1. *Proof of* 18 *from* G, 10.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
By G, ABCY . ⊃ . BCYA.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX.
Hence by G, ABXY ∪ BCYX.

THEOREM 18. 2. *Proof of* 18 *from* 12, 14.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY is false.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ABXY ∪ BCYX.

THEOREM 18. 3. *Proof of* 18 *from* G, 12, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
By G, ABCX . ⊃ . CXAB. By G, ABCY . ⊃ . CYAB.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. By G, CXYA ∪ CYXA.

Case 1. If CXYA, by 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.

Case 2. If CYXA, by 12, CYXA . CYAB . ⊃ . YXAB. By G, ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ABXY ∪ BCYX.

THEOREM 18. 4. *Proof of* 18 *from* G, 12, 16.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose BCYX is false.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY.
By G, ABCY . ⊃ . BCYA. By G, ABCX . ⊃ . CXAB.
By 12, BCXY . BCYA . ⊃ . CXYA.
By 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.
Therefore ABXY ∪ BCYX.

THEOREM 18. 5. *Proof of* 18 *from* G, 12, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY is false.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX. By G, CYXA.
By G, ABCY . ⊃ . CYAB.
By 12, CYXA . CYAB . ⊃ . YXAB. By G, ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ABXY ∪ BCYX.

THEOREM 18. 6. *Proof of* 18 *from* G, 12, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose BCYX is false. By G, ABCX . ⊃ . CXAB.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY. By G, CXYA.
By 12, CXYA . CXAB . ⊃ . XYAB. By G, ABXY.
Therefore ABXY ∪ BCYX.

THEOREM 18. 7. *Proof of* 18 *from* G, 13, 14.

THEOREM 18. 8. *Proof of* 18 *from* G, 13, 15.

THEOREM 18. 9. *Proof of* 18 *from* G, 13, 16.

THEOREM 18. 10. *Proof of* 18 *from* G, 13, 17.

THEOREM 18. 11. *Proof of* 18 *from* G, 13, 19.

These five theorems follow from Theorems 18. 2 to 18. 6, by aid of Theorem 12. 2.

THEOREM 18. 12. *Proof of* 18 *from* G, 11, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX.

Case 1. If ACXY, by G, YACX. By G, ABCY . ⊃ . YABC.
By 11, YABC . YACX . ⊃ . YABX. By G, ABXY.

Case 2. If ACYX, by G, CYXA. By G, ABCY . ⊃ . CYAB.
By 11, CYXA . CYAB . ⊃ . CYXB. By G, BCYX.
Therefore ABXY ∪ BCYX.

THEOREM 18. 13. *Proof of* 18 *from* G, 11, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY is false. By G, ABCY . ⊃ . CYAB.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX. By G, CYXA.
By 11, CYXA . CYAB . ⊃ . CYXB. By G, BCYX.
Therefore ABXY ∪ BCYX.

THEOREM 18. 14. *Proof of* 18 *from* G, 11, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose BCYX is false. By G, ABCY . ⊃ . YABC.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY. By G, YACX.
By 11, YABC . YACX . ⊃ . YABX. By G, ABXY.
Therefore ABXY ∪ BCYX.

THEOREM 18. 15. *Proof of* 18 *from* G, H, 14, 15.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY = 0 and BCYX = 0. By G, ABCY . ⊃ . YABC.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX. By G, XABY.
By G, ABCX . ⊃ . XABC.
By 14, XABY . XABC . ⊃ . XAYC ∪ XACY. By G, CXAY ∪ ACYX.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. By G, CXYA ∪ ACYX.
Now if ACYX = 0, then CXAY and CXYA, which is impossible, by H.
Hence ACYX. By G, YXAC. By G, ABYX . ⊃ . YXAB.
By 14, YXAB . YXAC . ⊃ . YXBC ∪ YXCB. By G, BCYX ∪ BYXC.
But BCYX = 0. Hence BYXC. By G, ABYX . ⊃ . BYXA.

By 14, BYXC . BYXA . ⊃ . BYCA ∨ BYAC. By G, ABYC ∨ YACB.
But ABYC . vs . ABCY, by H; and YACB . vs . YABC, by H.
Therefore ABXY ∨ BCYX.

THEOREM 18. 16. *Proof of* 18 *from* G, H, 15, 16.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ BCYX.
Suppose ABXY = o and BCYX = o.
By 16, ABCX . ABCY . ⊃ . BCXY ∨ BCYX. Hence BCXY.
By G, ABCX . ⊃ . BCXA.
By 16, BCXY . BCXA . ⊃ . CXYA ∨ CXAY. By G, ACXY ∨ XAYC.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. By G, ACXY ∨ XACY.
Now if ACXY = o, then XAYC and XACY, which is impossible, by H.
Hence ACXY. By G, CXYA. By G, BCXY . ⊃ . CXYB.
By 16, CXYA . CXYB . ⊃ . XYAB ∨ XYBA. By G, ABXY ∨ XYBA.
But ABXY = o. Hence XYBA. By G, BCXY . ⊃ . XYBC.
By G, ABCX . ⊃ . CXAB. By G, ABCX . ⊃ . BCAX.
By 15, XYBA . XYBC . ⊃ . XBAC ∨ XBCA. By G, CXBA ∨ BCAX.
But CXBA . vs . CXAB, by H, and BCAX . vs . BCXA, by H.
Therefore ABXY ∨ BCYX.

THEOREM 18. 17. *Proof of* 18 *from* G, H, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ BCYX.

Suppose ABXY is false. By G, ABCY . ⊃ . YABC.
By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. Hence ACYX. By G, YXAC.
By 17, ABCY . ABCX . ⊃ . ABYX ∨ ACXY.
But ACXY . vs . ACYX, by H. Hence ABYX. By G, YXAB.
By 17, YXAB . YXAC . ⊃ . YXBC ∨ YACB.
But YACB . vs . YABC, by H. Hence YXBC. By G, BCYX.
Therefore ABXY ∨ BCYX.

THEOREM 18. 18. *Proof of* 18 *from* G, H, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∨ BCYX.
Suppose BCYX is false. By G, ABCY . ⊃ . CYAB.
By 19, ABCX . ABCY . ⊃ . ACXY ∨ BCYX. Hence ACXY. By G, CXYA.
By 19, ABCY . ABCX . ⊃ . ACYX ∨ BCXY.
But ACYX . vs . ACXY, by H. Hence BCXY. By G, CXYB.
By 19, CXYB . CXYA . ⊃ . CYBA ∨ XYAB.
But CYBA . vs . CYAB, by H. Hence XYAB. By G, ABXY.
Therefore ABXY ∨ BCYX.

THEOREM 18. 19. *Proof of* 18 *from* H, 10, 14, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY = 0 and BCYX = 0.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY.
By 10, ABYX . C . ⊃ . ACYX ∪ ABYC.
But ACYX . vs . ACXY, by H; and ABYC . vs . ABCY, by H.
Therefore ABXY ∪ BCYX.

THEOREM 18. 20. *Proof of* 18 *from* H, 13, 14, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY = 0 and BCYX = 0.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX. Hence ABYX.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY.
By 13, ABCY . ABYX . ⊃ . ACYX. But ACYX . vs . ACXY.
Therefore ABXY ∪ BCYX.

THEOREM 18. 21. *Proof of* 18 *from* H, 17, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY = 0 and BCYX = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY.
But ACYX . vs . ACXY, by H.
Therefore ABXY ∪ BCYX.

THEOREM 18. 22. *Proof of* 18 *from* H, 12, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY is false.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. But ACXY . vs . ACYX, by H.
Hence ABYX. By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ABXY ∪ BCYX.

THEOREM 18. 23. *Proof of* 18 *from* R, G, 11, 14.

THEOREM 18. 24. *Proof of* 18 *from* R, G, 11, 16.

THEOREM 18. 25. *Proof of* 18 *from* R, G, H, 14.

THEOREM 18. 26. *Proof of* 18 *from* R, G, H, 16.

THEOREM 18. 27. *Proof of* 18 *from* R, F, G, 12.

THEOREM 18. 28. *Proof of* 18 *from* R, F, G, 13.

These six theorems follow from Theorems 15. 25 to 15. 26 and 15. 33 to 15. 36 (which give Postulate 15), by the aid of Theorem 18. 12.

THEOREM 18. 29. *Proof of* 18 *from* R, 10, 13, 14, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY = 0 and BCYX = 0. By R, XYCB = 0.
By 14, ABCX . ABCY ⊃ ABXY ∪ ABYX. Hence ABYX.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY.
By 13, ABCY . ABYX . ⊃ . ACYX. By R, XYCA.
By 10, XYCA . B . ⊃ . XBCA ∪ XYCB. Hence XBCA. By R, ACBX.
By 13, ACBX . ACXY . ⊃ . ABXY, which is false.
Therefore ABXY ∪ BCYX.

THEOREM 18. 30. *Proof of* 18 *from* R, 10, 12, 17.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
In view of Theorem 11. 6, Theorem 16. 31, and Theorem 19. 6, we may make use of Postulates 11, 16, and 19, as well as R, 10, 12, 17, in this proof.
Suppose ABXY = 0 and BCYX = 0. By R, YXBA = 0 and XYCB = 0.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. Hence ACYX. By R, XYCA.
By 19, ABCX . ABCY . ⊃ . ACXY ∪ BCYX. Hence ACXY. By R, YXCA.
By 10, XYCA . B . ⊃ . XBCA ∪ XYCB. Hence XBCA. By R, ACBX.
By 11, ACBX . ACXY . ⊃ . ACBY.
By 16, ACBX . ACBY . ⊃ . CBXY ∪ CBYX. By R, YXBC ∪ XYBC.
But if YXBC, then by 11, YXBC . YXCA . ⊃ . YXBA, which is false. Hence XYBC.
By 11, XYBC . XYCA . ⊃ . XYBA. By R, ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX, which is false.
Therefore ABXY ∪ BCYX.

THEOREM 18. 31. *Proof of* 18 *from* R, 10, 13, 17, 19.

To prove: ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
Suppose ABXY = 0 and BCYX = 0.
By R, XYCB = 0.

By 17, ABCX . ABCY . ⊃ . ABXY ∨ ACYX. But ABXY = 0. Hence ACYX. By R, XYCA.
By 10, XYCA . B . ⊃ . XBCA ∨ XYCB. But XYCB = 0. Hence XBCA. By R, ACBX.
By 13, if ACXY, then ACBX . ACXY . ⊃ . ABXY, which = 0. Hence ACXY = 0.
By 19, ABCX . ABCY . ⊃ . ACXY ∨ BCYX. But ACXY = 0 and BCYX = 0. Hence the supposition is false. Therefore ABXY ∨ BCYX.

Forty Proofs of Postulate 19.

Theorem 19. 1. *Proof of* 19 *from* G, H, 10.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
Suppose BCYX is false. By G, ABCY . ⊃ . BCYA.
By 10, BCYA . X . ⊃ . BXYA ∨ BCYX. Hence BXYA. By G, ABXY.
By 10, ABXY . C . ⊃ . ACXY ∨ ABXC.
But ABXC . vs . ABCX, by H. Hence ACXY.
Therefore ACXY ∨ BCYX.

Theorem 19. 2. *Proof of* 19 *from* G, 10, 12.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . BCYA.
By 10, BCXA . Y . ⊃ . BYXA ∨ BCXY. By G, ABYX ∨ BCXY.
If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX.
If BCXY, then by 12, BCXY . BCYA . ⊃ . CXYA. By G, ACXY.
Therefore ACXY ∨ BCYX.

Theorem 19. 3. *Proof of* 19 *from* G, 12, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
By G, ABCY . ⊃ . BCYA.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX.

Case 1. If ABXY, by 12, ABCX . ABXY . ⊃ . BCXY.
By 12, BCXY . BCYA . ⊃ . CXYA. By G, ACXY.

Case 2. If ABYX, by 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ACXY ∨ BCYX.

Theorem 19. 4. *Proof of* 19 *from* G, 12, 15.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
Suppose ACXY is false. By G, ABCY . ⊃ . CYAB.

By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. Hence ACYX. By G, CYXA.
By 12, CYXA . CYAB . ⊃ . YXAB. By G, ABYX.
By 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ACXY ∪ BCYX.

THEOREM 19. 5. *Proof of* 19 *from* G, 12, 16.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose BCYX is false. By G, ABCY . ⊃ . BCYA.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY.
By 12, BCXY . BCYA . ⊃ . CXYA. By G, ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 6. *Proof of* 19 *from* 12, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY.
If ABYX, by 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ACXY ∪ BCYX.

THEOREM 19. 7. *Proof of* 19 *from* G, 12, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By G, ABCY . ⊃ . BCYA.
By 18, ABCY . ABCX . ⊃ . ABYX ∪ BCXY.
If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX.
If BCXY, then by 12, BCXY . BCYA . ⊃ . CXYA. By G, ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 8. *Proof of* 19 *from* G, 10, 13.

THEOREM 19. 9. *Proof of* 19 *from* G, 13, 14.

THEOREM 19. 10. *Proof of* 19 *from* G, 13, 15.

THEOREM 19. 11. *Proof of* 19 *from* G, 13, 16.

THEOREM 19. 12. *Proof of* 19 *from* G, 13, 17.

These five theorems follow from Theorems 19. 2 to 19. 6 by aid of Theorem 12. 2.

THEOREM 19. 13. *Proof of* 19 *from* 13, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By 13, ABCX . ABCY . ⊃ . ABXY ∪ BCYX.
If ABXY, then by 13, ABCX . ABXY . ⊃ . ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 14. *Proof of* 19 *from* G, 11, 15.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose ACXY is false. By G, ABCY . ⊃ . CYAB.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. Hence ACYX. By G, CYXA.
By 11, CYXA . CYAB . ⊃ . CYXB. By G, BCYX.
Therefore ACXY ∪ BCYX.

THEOREM 19. 15. *Proof of* 19 *from* G, 10, 11, 16.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose BCYX is false. By G, ABCY . ⊃ . BCYA.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY. By G, XYBC.
By 10, BCYA . X . ⊃ . BXYA ∪ BCYX. Hence BXYA. By G, XYAB.
By 11, XYAB . XYBC . ⊃ . XYAC. By G, ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 16. *Proof of* 19 *from* G, 11, 16, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose ACXY = 0, and BCYX = 0. By G, ABCY . ⊃ . CYAB.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY. By G, XYBC.
By 17, ABCX . ABCY . ⊃ . ABXY ∪ ACYX. By G, XYAB ∪ CYXA.

Case 1. If XYAB, by 11, XYAB . XYBC . ⊃ . XYAC. By G, ACXY.

Case 2. If CYXA, by 11, CYXA . CYAB . ⊃ . CYXB. By G, BCYX.
Therefore ACXY ∪ BCYX.

THEOREM 19. 17. *Proof of* 19 *from* G, 11, 16, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose BCYX = 0. By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY. By G, XYBC.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence ABXY. By G, XYAB.
By 11, XYAB . XYBC . ⊃ . XYAC. By G, ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 18. *Proof of* 19 *from* G, H, 14, 15.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
Suppose ACXY = 0 and BCYX = 0. By G, ABCY . ⊃ . YABC.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. Hence ACYX.
By G, CYXA. By G, YXAC.
By 14, ABCX . ABCY . ⊃ . ABXY ∨ ABYX. By G, YABX ∨ YXAB.

Case 1. Suppose YABX. By 14, YABX . YABC . ⊃ . YAXC ∨ YACX.
By G, CYAX ∨ ACXY. But CYAX . vs . CYXA by H; and ACXY = 0.

Case 2. Suppose YXAB. By 14, YXAB . YXAC . ⊃ . YXBC ∨ YXCB.
By G, BCYX ∨ BYXC. But BCYX = 0. Hence BYXC.
By G, YXAB . ⊃ . BYXA. By G, ABCX . ⊃ . XABC.
By 15, BYXC . BYXA . ⊃ . BXCA ∨ BXAC. By G, ABXC ∨ XACB.
But ABXC . vs . ABCX, by H; and XACB . vs . XABC by H.
Therefore ACXY ∨ BCYX.

THEOREM 19. 19. *Proof of* 19 *from* G, H, 15, 16.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
Suppose ACXY = 0 and BCYX = 0. By G, ABCX . ⊃ . BCXA.
By 15, ABCX . ABCY . ⊃ . ACXY ∨ ACYX. Hence ACYX. By G, XACY.
By 16, ABCX . ABCY . ⊃ . BCXY ∨ BCYX. Hence BCXY.
By 16, BCXA . BCXY . ⊃ . CXAY ∨ CXYA. By G, XAYC ∨ ACXY.
But XAYC . vs . XACY, by H; and ACXY = 0.
Therefore ACXY ∨ BCYX.

THEOREM 19. 20. *Proof of* 19 *from* G, H, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∨ BCYX.
Suppose ACXY = 0 and BCYX = 0. By G, ABCX . ⊃ . XABC.
By 17, ABCY . ABCX . ⊃ . ABYX ∨ ACXY. Hence ABYX. By G, XABY.
By 17, XABY . XABC . ⊃ . XAYC ∨ XBCY. By G, AYCX ∨ BCYX.
Hence AYCX. By G, CXAY. By G, ABCX . ⊃ . CXAB.
By 17, CXAY . CXAB . ⊃ . CXYB ∨ CABY. By G, BCXY ∨ ABYC.
But ABYC . vs . ABCY, by H. Hence BCXY.
By G, ABCX . ⊃ . BCXA. By G, ABCY . ⊃ . BCYA.
By 17, BCXA . BCXY . ⊃ . BCAY ∨ BXYA.
But BCAY . vs . BCYA, by H. Hence BXYA. By G, ABXY.
But ABXY . vs . ABYX by H.
Therefore ACXY ∨ BCYX.

THEOREM 19. 21. *Proof of* 19 *from* G, H, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose BCYX = 0. By G, ABCY . ⊃ . YABC.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence ABXY. By G, YABX.
By 18, YABC . YABX . ⊃ . YACX ∪ ABXC.
But ABXC . vs . ABCX, by H. Hence YACX. By G, ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 22. *Proof of* 19 *from* H, 10, 12, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.

Case 1. If ABXY, then by 10, ABXY . C . ⊃ . ACXY ∪ ABXC.
But ABXC . vs . ABCX, by H. Hence ACXY.

Case 2. If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ACXY ∪ BCYX.

THEOREM 19. 23. *Proof of* 19 *from* H, 10, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose BCYX = 0.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence ABXY.
By 10, ABXY . C . ⊃ . ACXY ∪ ABYC.
But ABYC . vs . ABCY, by H. Hence ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 24. *Proof of* 19 *from* H, 17, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose ACXY = 0 and BCYX = 0.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence ABXY.
By 17, ABCY . ABCX . ⊃ . ABYX ∪ ACXY. Hence ABYX.
But ABXY . vs . ABYX, by H.
Therefore ACXY ∪ BCYX.

THEOREM 19. 25. *Proof of* 19 *from* 12, 13, 14.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By 14, ABCX . ABCY . ⊃ . ABXY ∪ ABYX.
If ABXY, then by 13, ABCX . ABXY . ⊃ . ACXY.
If ABYX, then by 12, ABCY . ABYX . ⊃ . BCYX.
Therefore ACXY ∪ BCYX.

THEOREM 19. 26. *Proof of* 19 *from* R, G, 17.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
By R, G, ABCX . ⊃ . CBAX. By R, G, ABCY . ⊃ . CBAY.
By 17, CBAX . CBAY . ⊃ . CBXY ∪ CAYX. By R, G, BCYX ∪ ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 27. *Proof of* 19 *from* R, G, 11, 14.

THEOREM 19. 28. *Proof of* 19 *from* R, G, 11, 16.

THEOREM 19. 29. *Proof of* 19 *from* R, G, H, 14.

THEOREM 19. 30. *Proof of* 19 *from* R, G, H, 16.

THEOREM 19. 31. *Proof of* 19 *from* R, F, G, 12.

THEOREM 19. 32. *Proof of* 19 *from* R, F, G, 13.

These six theorems follow from Theorems 15. 25 to 15. 26 and 15. 33 to 15. 36 (which give Postulate 15), by the aid of Theorems 19. 14, 19. 18, 19. 19, 19. 4 and 19. 10.

THEOREM 19. 33. *Proof of* 19 *from* R, H, 11, 18.

Proof: 13 from R, H, 11, 18, by Theorem 13. 11; then 19 from 13, 18, by Theorem 19. 13.

THEOREM 19. 34. *Proof of* 19 *from* R, 11, 12, 14.

Proof: 17 from R, 11, 12, 14 by Theorem 17. 35; then 19 from 12, 17, by Theorem 19. 6.

THEOREM 19. 35. *Proof of* 19 *from* R, 11, 12, 18.

Proof: 13 from R, 11, 12, by Theorem 13. 12; then 19 from 13, 18, by Theorem 19. 13.

THEOREM 19. 36. *Proof of* 19 *from* R, 11, 16, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose BCYX = 0.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY. By R, YXCB.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. Hence ABXY. By R, YXBA.
By 11, YXCB . YXBA . ⊃ . YXCA. By R, ACXY.
Therefore ACXY ∪ BCYX.

THEOREM 19. 37. *Proof of* 19 *from* R, H, 10, 12, 16.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose ACXY = 0 and BCYX = 0. By R, YXCA = 0 and XYCB = 0.
By 16, ABCX . ABCY . ⊃ . BCXY ∪ BCYX. Hence BCXY. By R, YXCB.
By 10, YXCB . A . ⊃ . YACB ∪ YXCA. Hence YACB. By R, BCAY.
By R, ABCX . ⊃ . XCBA. By R, ABCY . ⊃ . YCBA.
Suppose ABYX; then, by 13, ABCY . ABYX . ⊃ . BCYX. Hence ABYX = 0.
By R, XYBA = 0.
By 10, XCBA . Y . ⊃ . XYBA ∪ XCBY. Hence XCBY.
By H, ABCX . ABXC = 0. .
Suppose ABXY; then by 10, ABXY . C . ⊃ . ACXY ∪ ABXC.
Hence ABXY = 0.
By R, YXBA = 0.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. Hence YCBX. By R, XBCY.
By H, YCBA . VS . YCAB. By R, BACY = 0.
Suppose XBAC; then by 12, XBAC . XBCY . ⊃ . BACY. Hence XBAC = 0.
By H, BCAY . VS . BCYA. By H, YACB . VS . YABC. By R, CBAY = 0.
Suppose XBYA; then by 12, XBCY . XBYA . ⊃ . BCYA. Hence XBYA = 0.
By R, AYBX = 0.
By 16, XCBA . XCBY . ⊃ . CBAY ∪ CBYA. Hence CBYA. By R, AYBC.
By 10, AYBC . X . ⊃ . AXBC ∪ AYBX. Hence AXBC. By R, CBXA.
By H, YXCB . ⊃ . YXBC = 0. By R, CBXY = 0.
By 10, CBXA . Y . ⊃ . CYXA ∪ CBXY. Hence CYXA. By R, AXYC.
Suppose AXCB; then by 12, AXYC . AXCB . ⊃ . XYCB. Hence AXCB = 0.
By R, BCXA = 0.
By 10, BCXY . A . ⊃ . BAXY ∪ BCXA. Hence BAXY. By R, YXAB.
By H, CBXA . ⊃ . CBAX = 0.
Suppose YCAX; then by 12, YCBA . YCAX . ⊃ . CBAX. Hence YCAX = 0.
By R, XACY = 0.
By H, YCBA . ⊃ . YCAB = 0.
By 10, YXAB . C . ⊃ . YCAB ∪ YXAC. Hence YXAC.
By 12, YXAC . YXCB . ⊃ . XACB.
By 10, XACB . Y . ⊃ . XYCB ∪ XACY. But XACY = 0, and XYCB = 0.
Therefore ACXY ∪ BCYX.

THEOREM 19. 38. *Proof of* 19 *from* R, 10, 12, 15.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose ACXY = 0 and BCYX = 0. By R, YXCA = 0 and XYCB = 0.
By 15, ABCX . ABCY . ⊃ . ACXY ∪ ACYX. Hence ACYX. By R, XYCA.
By 10, XYCA . B . ⊃ . XBCA ∪ XYCB. Hence XBCA. By R, ACBX.

By R, ABCX . ⊃ . XCBA. By R, ABCY . ⊃ . YCBA.
Now, ABYX = o (for if not, by 12, ABCY . ABYX . ⊃ . BCYX). By R, XYBA = o.
By 10, XCBA . Y . ⊃ . XYBA ∪ XCBY. Hence XCBY. By R, YBCX.
By 10, ACYX . B . ⊃ . ABYX ∪ ACYB. Hence ACYB. By R, BYCA.
Now BYXC = o (for if not, by 12, BYXC . BYCA . ⊃ . YXCA). By R, CXYB = o.
By 12, ACYB . ACBX . ⊃ . CYBX. By R, XBYC.

Case 1. Suppose BYCX. By R, XCYB.
By 12, XCYB . XCBA . ⊃ . CYBA. By R, ABYC.
By 10, ABYC . X . ⊃ . AXYC ∪ ABYX. Hence AXYC. By R, CYXA.
By 12, CYBX . CYXA . ⊃ . YBXA.
By 12, YBCX . YBXA . ⊃ . BCXA. By R, AXCB.
By 12, AXYC . AXCB . ⊃ . XYCB. But XYCB = o.
Therefore BYCX = o. By R, XCYB = o.
By 10, BYCA . X . ⊃ . BXCA ∪ BYCX. Hence BXCA. By R, ACXB.
By 10, ACXB . Y . ⊃ . AYXB ∪ ACXY. Hence AYXB. By R, BXYA.
By 12, ACYX . ACXB . ⊃ . CYXB.
Now ABYC = o (for if not, by 12, ABYC . ABCX . ⊃ . BYCX). By R, CYBA = o.
By 10, CYBX . A . ⊃ . CABX ∪ CYBA. Hence CABX. By R, XBAC.

Case 2. Suppose ACBY.
By 12, ACBY . ACYX . ⊃ . CBYX. By R, XYBC.
By 12, XYBC . XYCA . ⊃ . YBCA. By R, ACBY.
By 15, ACBX . ACBY . ⊃ . ABXY ∪ ABYX. Hence ABXY.
By 10, ABXY . C . ⊃ . ACXY ∪ ABXC. Hence ABXC.
By 10, XYBC . A . ⊃ . XABC ∪ XYBA. Hence XABC. By R, CBAX.
By 12, ABXC . ABCY . ⊃ . BXCY.
By 12, BXCY . BXYA . ⊃ . XCYA.
By 12, XCBY . XCYA . ⊃ . CBYA.
By 12, CBYA . CBAX . ⊃ . BYAX. By R, XAYB.
By 10, XAYB . C . ⊃ . XCYB ∪ XAYC. Hence XAYC. By R, CYAX.
By 12, CYAX . CYXB . ⊃ . YAXB. By R, BXAY.
By 12, BXCA . BXAY . ⊃ . XCAY.
By 12, XCBA . XCAY . ⊃ . CBAY. By R, YABC.
By 12, YAXB . YABC . ⊃ . AXBC.
Now CYXA is either true or false.
If CYXA, then by R, AXYC,
and by 12, CYBX . CYXA . ⊃ . YBXA;

then, by 12, YBCX . YBXA . ⊃ . BCXA; then by R, AXCB;
then by 12, AXYC . AXCB . ⊃ . XYCB, which is false.
If CYXA = 0, then by R, AXYC = 0;
then by 15, ABXY . ABXC . ⊃ . AXYC ∪ AXCY; then AXCY;
whence by R, YCXA;
then by 12, AXBC . AXCY . ⊃ . XBCY; then by R, YCBX;
then by 12, YCBX . YCXA . ⊃ . CBXA;
then by 12, CBYX . CBXA . ⊃ . BYXA; then by R, AXYB;
then by 12, AXCY . AXYB . ⊃ . XCYB, which is false.
Therefore ACBY = 0. By R, YBCA = 0.
By 10, ACBX . Y . ⊃ . ACBY ∪ AYBX. Hence AYBX. By R, XBYA.
By 10, YBCX . A . ⊃ . YACX ∪ YBCA. Hence YACX. By R, XCAY.
Now XYBC = 0 (for if not, by 12, XYBC . XYCA . ⊃ . YBCA).
By 12, XCBA . XCAY . ⊃ . CBAY.
By 12, XBYA . XBAC . ⊃ . BYAC. By R, CAYB.
By 10, CAYB . X . ⊃ . CXYB ∪ CAYX. Hence CAYX.
Now CABY = 0 (for if not, by 12, CABY . CAYX . ⊃ . ABYX). By R, YBAC.
Then XCAB = 0 (for if not, by 12, XCAB . XCBY . ⊃ . CABY). By R, BACX = 0.
By 10, XCAY . B . ⊃ . XBAY ∪ XCAB. Hence XBAY.
By 12, XBCA . XBAY . ⊃ . BCAY. By R, YACB.
By 12, XBAY . XBYC . ⊃ . BAYC.
Now BAYX = 0 (for if not, by 15, BAYX . BAYC . ⊃ . BYXC ∪ BYCX).

Case 3. Suppose XBCY = 0. By R, YCBX = 0.
By 10, YCBA . X . ⊃ . YXBA ∪ YCBX. Hence YXBA.
By 15, YACX . YACB . ⊃ . YCXB ∪ YCBX. Hence YCXB. By R, BXCY.
By 12, BXCY . BXYA . ⊃ . XCYA.
By 12, XCBY . XCYA . ⊃ . CBYA. By R, AYBC.
By 12, AYXB . AYBC . ⊃ . YXBC.
By 15, YXBA . YXBC . ⊃ . YBAC ∪ YBCA.
But YBAC = 0 and YBCA = 0.
Therefore XBCY.
By 12, XBCY . XBYA . ⊃ . BCYA.
By 12, XBAC . XBCY . ⊃ . BACY.
By 10, BACY . X . ⊃ . BXCY ∪ BACX. Hence BXCY.

Case 4. Suppose CXAB = 0.
By 10, CYAB . X . ⊃ . CXAB ∪ CYAX. Hence CYAX. By R, XAYC.
By 10, BYAC . X . ⊃ . BXAC ∪ BYAX.

If BYAX, then by R, XAYB;
then by 15, XAYB . XAYC . ⊃ . XYBC ∪ XYCB;
where XYBC = 0 and XYCB = 0.
Hence BXAC.
By 12, BXAC . BXCY . ⊃ . XACY; by R, YCAX.
By 12, YCBA . YCAX . ⊃ . CBAX. By R, XABC.
By 10, XABC . Y . ⊃ . XYBC ∪ XABY. Hence XABY.
By 12, XABY . XAYC . ⊃ . ABYC, which is false.
Therefore, CXAB. By R, BAXC.
By 10, BAXC . Y . ⊃ . BYXC ∪ BAXY. Hence BAXY. By R, YXAB.
By 12, BAXY . BAYC . ⊃ . AXYC.
By 10, AXYC . B . ⊃ . ABYC ∪ AXYB. Hence AXYB. By R, BYXA.
By 12, BYXA . BYAC . ⊃ . YXAC. By R, CAXY.
By 12, CABY . CAXY . ⊃ . ABXY.
By 12, ABCX . ABXY . ⊃ . BCXY.
By 12, BCXY . BCYA . ⊃ . CXYA.
By 12, CXYA . CXAB . ⊃ . XYAB. By R, BAYX.
But BAYX = 0.
Therefore ACXY ∪ BCYX.

THEOREM 19. 39. *Proof of* 19 *from* R, 10, 12, 14.

Proof: Postulate 15 follows from R, 10, 12, 14, by Theorem 15. 45.
Then 19 follows from R, 10, 12, 15, by Theorem 19. 38.
Or thus: Postulate 18 follows from 12 and 14 by Theorem 18. 2. Then 19 follows from R, 10, 12, 18 by Theorem 19. 40 (below).

THEOREM 19. 40. *Proof of* 19 *from* R, 10, 12, 18.

To prove: ABCX . ABCY . ⊃ . ACXY ∪ BCYX.
Suppose both ACXY = 0 and BCYX = 0. By R, YXCA = 0, XYCB = 0.
By R, ABCX, XCBA; ABCY, YCBA.
By 18, ABCX . ABCY . ⊃ . ABXY ∪ BCYX. But BCYX = 0. Hence ABXY. By R, YXBA.
By 12, ABCX . ABXY . ⊃ . BCXY. By R, YXCB.
By 10, YXCB . A . ⊃ . YACB ∪ YXCA. But YXCA = 0. Hence YACB. By R, BCAY.
By 10, ABXY . C . ⊃ . ACXY ∪ ABXC. But ACXY = 0. Hence ABXC. By R, CXBA.
By 12, ABXC . ABCY . ⊃ . BXCY. By R, YCXB.
By 12, if ABYX, then ABCY . ABYX . ⊃ . BCYX, which = 0.
Hence ABYX = 0. By R, XYBA = 0.

By 12, if CXYB, then CXYB . CXBA . ⊃ . XYBA, which = 0.
Hence CXYB = 0. By R, BYXC = 0.
By 10, XCBA . Y . ⊃ . XYBA ∪ XCBY. But XYBA = 0. Hence XCBY.
By R, YBCX.
By 10, BCAY . X . ⊃ . BXAY ∪ BCAX.
By 10, YCXB . A . ⊃ . YAXB ∪ YCXA. By R, BXAY ∪ YCXA.
Hence BXAY ∪ (BCAX . YCXA).

Supposition 1. Suppose BXAY = 0. By R, YAXB = 0.
Then BCAX and YCXA. By R, XACB and AXCY.
By 10, XACB . Y . ⊃ . XYCB ∪ XACY. But XYCB = 0. Hence XACY.
By R, YCAX.
By 12, YCBA . YCAX . ⊃ . CBAX.
By 12, YCAX . YCXB . ⊃ . CAXB. By R, BXAC.
By 10, BXAC . Y . ⊃ . BYAC ∪ BXAY. But BXAY = 0. Hence BYAC.
By R, CAYB.
By 18, BCAY . BCAX . ⊃ . BCYX ∪ CAXY. But BCYX = 0. Hence CAXY.
By 12, CAXY . CAYB . ⊃ . AXYB. By R, BYXA.
By 12, AXCY . AXYB . ⊃ . XCYB.
By 12, XCYB . XCBA . ⊃ . CYBA. By R, ABYC.
By 10, ABYC . X . ⊃ . AXYC ∪ ABYX. But ABYX = 0. Hence AXYC.
By 10, BYXA . C . ⊃ . BCXA ∪ BYXC. But BYXC = 0. Hence BCXA.
By R, AXCB.
By 12, AXYC . AXCB . ⊃ . XYCB, which = 0.
Hence Supposition 1 is false. Therefore, BXAY, and by R, YAXB.

Supposition 2. Suppose XCAY = 0. By R, YACX = 0.
By 12, if BXCA, then BXCA . BXAY . ⊃ . XCAY, which = 0. Hence BXCA = 0.
By 10, YBCX . A . ⊃ . YACX ∪ YBCA. But YACX = 0. Hence YBCA. By R, ACBY.
By 12, if YBXC, then YBXC . YBCA . ⊃ . BXCA, which = 0. Hence YBXC = 0. By R, CXBY = 0.
By 10, CXBA . Y . ⊃ . CYBA ∪ CXBY. But CXBY = 0. Hence CYBA. By R, ABYC.
By 12, ABXY . ABYC . ⊃ . BXYC.
By 12, BXAY . BXYC . ⊃ . XAYC. By R, CYAX.
By 12, CYBA . CYAX . ⊃ . YBAX.
By 10, ABYC . X . ⊃ . AXYC ∪ ABYX. But ABYX = 0. Hence AXYC.
By 12, if AXCB, then AXYC . AXCB . ⊃ . XYCB, which = 0. Hence AXCB = 0. By R, BCXA = 0.
By 10, BCXY . A . ⊃ . BAXY ∪ BCXA. But BCXA = 0. Hence BAXY.

By 12, if BACX, then BACX . BAXY . ⊃ . ACXY, which = 0. Hence BACX = 0.
By 12, if YBAC, then YBAC . YBCX . ⊃ . BACX, which = 0. Hence YBAC = 0.
By 12, if YBXA, then YBCX . YBXA . ⊃ . BCXA, which = 0. Hence YBXA = 0. By R, AXBY = 0.
By 12, if CYBX, then CYBX . CYXA . ⊃ . YBXA, which = 0. Hence CYBX = 0. By R, XBYC = 0.
By 10, ACBY . X . ⊃ . AXBY ∪ ACBX. But AXBY = 0. Hence ACBX.
By 12, if ACYB, then ACYB . ACBX . ⊃ . CYBX, which = 0. Hence ACYB = 0.
By 10, XAYC . B . ⊃ . XBYC ∪ XAYB. But XBYC = 0. Hence XAYB.
By 12, if XACY, then XACY . XAYB . ⊃ . ACYB, which = 0. Hence XACY = 0. By R, YCAX = 0.
By 10, YBAX . C . ⊃ . YCAX ∪ YBAC, both of which = 0.
Hence Supposition 2 is false. Hence XCAY. By R, YACX.
We have BXAY and XCAY. By R, YAXB and YACX.
By 12, YACX . YAXB . ⊃ . ACXB.
By 10, ACXB . Y . ⊃ . AYXB ∪ ACXY. But ACXY = 0. Hence AYXB. By R, BXYA.
By 12, BXCY . BXYA . ⊃ . XCYA.
By 12, XCBY . XCYA . ⊃ . CBYA. By R, AYBC.
By 12, AYXB . AYBC . ⊃ . YXBC. By R, CBXY.
By 18, YXBC . YXBA . ⊃ . YXCA ∪ XBAC. But YXCA = 0. Hence XBAC. By R, CABX.

Supposition 3. Suppose BAXY = 0. By R, YXAB = 0.
By 12, if CBAX, then CBAX . CBXY . ⊃ . BAXY, which = 0. Hence CBAX = 0.
By 12, if YCAX, then YCBA . YCAX . ⊃ . CBAX, which = 0. Hence YCAX = 0. By R, XACY = 0.
By 12, if BXAC, then BXAC . BXCY . ⊃ . XACY, which = 0. Hence BXAC = 0. By R, CAXB = 0.
By 10, BCXY . A . ⊃ . BAXY ∪ BCXA. But BAXY = 0. Hence BCXA. By R, AXCB.
By 12, if AXYC, then AXYC . AXCB . ⊃ . XYCB, which = 0. Hence AXYC = 0. By R, CYXA = 0.
By 10, if XACB, then XACB . Y . ⊃ . XYCB ∪ XACY. But XYCB = 0 and XACY = 0. Hence XACB = 0.
By 12, if YXAC, then YXAC . YXCB . ⊃ . XACB, which = 0. Hence YXAC = 0.

By 18, YXBA . YXBC . ⊃ . YXAC ∪ XBCA. But YXAC = 0. Hence XBCA.
By 10, if YCAB, then YCAB . X . ⊃ . YXAB ∪ YCAX. But YXAB = 0 and YCAX = 0. Hence YCAB = 0. By R, BACY = 0.
By 12, if XBCY, then XBAC . XBCY . ⊃ . BACY, which = 0. Hence XBCY = 0.
By 10, XBCA . Y . ⊃ . XYCA ∪ XBCY. But XBCY = 0. Hence XYCA. By R, ACYX.
By 12, ACYX . ACXB . ⊃ . CYXB.
By 10, CYXB . A . ⊃ . CAXB ∪ CYXA, both of which = 0.
Hence Supposition 3 is false. Hence BAXY. By R, YXAB.

The items used in the remainder of the proof are the following, with their reverses: ACXY = 0, XYCB = 0; XCBA, ABCY; BCXY; YACB; YCXB; ABYX = 0; BYXC = 0; YBCX; BXAY; XCAY; CBYA, YXBC; CABX; BAXY.
By 12, if BACX, then BACX . BAXY . ⊃ . ACXY, which = 0. Hence BACX = 0. By R, XCAB = 0.
By 10, XCAY . B . ⊃ . XBAY ∪ XCAB. But XCAB = 0. Hence XBAY. By R, YABX.
By 12, if YBAC, then YBAC . YBCX . ⊃ . BACX, which = 0. Hence YBAC = 0. By R, CABY = 0.
By 10, CABX . Y . ⊃ . CYBX ∪ CABY. But CABY = 0. Hence CYBX. By R, XBYC.
By 12, XBAY . XBYC . ⊃ . BAYC.
By 12, BAXY . BAYC . ⊃ . AXYC.
By 12, YXAB . YXBC . ⊃ . XABC. By R, CBAX.
By 12, CBYA . CBAX . ⊃ . BYAX.
By 12, YACB . YABX . ⊃ . ACBX. By R, XBCA.
By 12, XBYC . XBCA . ⊃ . BYCA.
By 12, BYCA . BYAX . ⊃ . YCAX.
By 12, YCAX . YCXB . ⊃ . CAXB.
By 12, if AXCB, then AXYC . AXCB . ⊃ . XYCB, which = 0. Hence AXCB = 0. By R, BCXA = 0.
By 10, if BYXA, then BYXA . C . ⊃ . BCXA ∪ BYXC, both of which = 0. Hence BYXA = 0. By R, AXYB = 0.
By 18, if CAXY, then CAXB . CAXY . ⊃ . CABY ∪ AXYB, both of which = 0. Hence CAXY = 0. By R, YXAC = 0.
By 12, if BCAX, then BCAX . BCXY . ⊃ . CAXY, which = 0. Hence BCAX = 0.
By 10, BYAX . C . ⊃ . BCAX ∪ BYAC. But BCAX = 0. Hence BYAC.
By 12, if BYXA, then BYXA . BYAC . ⊃ . YXAC, which = 0. Hence BYXA = 0.

By 12, XCBA . XCAY . ⊃ . CBAY. By R, YABC.
By 12, YAXB . YABC . ⊃ . AXBC. By R, CBXA.
By 12, if CBYX, then CBYX . CBXA . ⊃ . BYXA, which = 0. Hence CBYX = 0. By R, XYBC = 0.
By 12, YXAB . YXBC . ⊃ . XABC. By R, CBAX.
By 12, CBYA . CBAX . ⊃ . BYAX.
By 10, XCAY . B . ⊃ . XBAY ∨ XCAB. But XCAB = 0. Hence XBAY. By R, YABX.
By 12, YACB . YABX . ⊃ . ACBX. By R, XBCA.
By 12, XBYC . XBCA . ⊃ . BYCA.
By 12, BYCA . BYAX . ⊃ . YCAX.
By 12, YCAX . YCXB . ⊃ . CAXB.
By 10, CAXB . Y . ⊃ . CYXB ∨ CAXY. But CAXY = 0. Hence CYXB. By R, BXYC.
By 12, BXAY . BXYC . ⊃ . XAYC.
By 10, XABC . Y . ⊃ . XYBC ∨ XABY. But XYBC = 0. Hence XABY.
By 12, XABY . XAYC . ⊃ . ABYC.
By 10, ABYC . X . ⊃ . AXYC ∨ ABYX. But ABYX = 0. Hence AXYC. By R, CYXA.
By 10, CABX . Y . ⊃ . CYBX ∨ CABY. But CABY = 0. Hence CYBX.
By 12, CYBX . CYXA . ⊃ . YBXA.
By 12, YBCX . YBXA . ⊃ . BCXA. By R, AXCB.
By 12, AXYC . AXCB . ⊃ . XYCB, which = 0.
Hence the original supposition is false. Hence ACXY ∨ BCYX.

Construction of Examples of Systems (K, R).

In the following examples of systems (K, R), the class K consists, in each case, of five elements, 1, 2, 3, 4, 5, and the relation R (A, B, C, D) is defined by exhibiting, in each case, the tetrads which are supposed to be true in that case. (There are 120 possible tetrads formed from the five elements, 1, 2, 3, 4, 5.)

For brevity, the notation 1234G denotes not only the tetrad 1234, but also all the tetrads which can be derived from it by the use of Postulate G; that is, 1234G stands for the four tetrads, 1234, 2341, 3412, 4123.

Similarly, 1234R denotes not only 1234 but also 4321 (which is derived from it by the use of Postulate R).

So 1234GR stands for all eight of the tetrads, 1234, 2341, 3412, 4123; 3214, 2143, 1432, 4321.

The numbering of the examples begins with 200 to avoid confusion with earlier lists.

The properties which these systems possess will be set forth in the tables below.

Ex. 200. 1234GR, 1235GR, 1245GR, 1345GR, 2345GR. ("Reversible order on a closed line.")

Ex. 201. All tetrads false.

Ex. 202. 1234G, 1235G, 1245G, 1345G, 2345G. ("Cyclic order.")

Ex. 203. 1234GR.

Ex. 204. 1234R, 1254R, 1324R, 1354R, 1524R, 1534R.

Ex. 205. 1234GR, 1235GR, 1243GR, 1245GR, 1253GR, 1254GR.

Ex. 206. All the ninety-six tetrads which include the element 1.

Ex. 207. 1234R, 1254R, 1324R, 1354R, 1524R, 1534R, 2135R, 2145R, 2315R, 2345R, 2415R, 2435R.

Ex. 208. All tetrads true.

Ex. 209. 1234R, 1235, 1245, 1345, 2145, 2345, 4521.

Ex. 210. 1234GR; 1235G, 1254G, 1324G, 1325G, 1354G, 1425G, 1435G.

Ex. 211. 1234GR, 1235G, 1245G, 1345G, 2345G.

Ex. 212. 1234R, 1235R, 1245R, 1345R, 2345R. ("Betweenness.")

Ex. 213. 1235GR, 1243GR, 1354GR, 1425GR, 2345GR.

Ex. 214. 1235GR, 1243GR, 1245GR, 1345GR, 2354GR.

Ex. 215. All the 120 possible tetrads *except* 2435GR.

Ex. 216. All except 1245R, 1354R, 2135R, 2453R, 3124R.

Ex. 217. 1234, 1245, 1254, 1324, 1325, 1345, 1354, 1524, 1534, 3125, 3145, 3215, 3245, 3415, 3425, with all their reverses.

Ex. 218. 1235R, 1243R, 1423R, 1425R, 1435R, 1543R, 4235R.

Ex. 219. 1234R, 1245R, 1354R, 2345R, 3125R.

Ex. 220. 1234R, 1245, 1345, 1534, 2145, 2345, 4521, 5213, 5413.

Ex. 221. 1234GR, 1235GR, 1243GR, 1245GR, 1253GR, 1254GR, 1345GR, 1354GR, 2345GR, 2354GR.

Ex. 222. 1234GR, 1235G, 1254G, 2345G, 3415G.

Ex. 223. All tetrads *except* 2135R, 2145R, 2315R, 2345R, 2415R, 2435R.

Ex. 224. 1234, 1235, 1245, 1324, 1325, 1345, 1423, 1425, 1435, 1523, 1542, 1543, 2143, 2153, 2154, 2314, 2315, 2354, 2413, 2453, 2513, 2514, 2543, 3154, 3245, 3514, 4215, 4235, 4315, 4325, with all their reverses.

Ex. 225. 1235R, 1243, 1254, 1354, 1435, 1543, 2145, 2345, 5421.

Ex. 226. 1234R, 1235R, 1254R, 1345R, 2145R, 2345R.

Ex. 227. 1234, 1235, 1243, 1245, 1253, 1254, 1345R, 1453R, 1543R, 2143, 2341, 2345, 2351, 2354, 2431, 2435, 2451, 2453, 2531, 2534, 2541, 2543, 3241, 3251, 4231, 4251, 4351, 4531, 5231, 5241, 5341.

Ex. 228. 1235, 1243, 1245, 1453, 2435R.

Ex. 229. 1235GR, 1243GR, 1345GR, 1425GR, 2345GR.

Ex. 230. 1345GR, 2345GR.

Ex. 231. 1234GR, 1235G, 4215G, 4315G, 5324G.

Ex. 232. 1234G, 1235GR, 1324GR, 1425GR, 1435GR, 2315G, 2345G, 2354GR, 4215G, 4315G.

Ex. 233. 1234G, 1235G, 1342G, 1345G, 1352G, 1354GR, 1423G, 1425GR, 1435GR, 1452G, 1523G, 1542G, 2345G, 2354G.

Ex. 234. All tetrads *except* 1245GR, 1254GR.

Ex. 235. 1234GR, 1235GR, 1324GR, 1325GR, 1425GR, 1435GR, 2345GR 2354GR.

Ex. 236. 1234GR, 1235GR, 1243GR, 1324GR, 1345GR, 1425GR, 2345GR, 2354GR, 2435GR.

Ex. 237. 1234GR, 1235GR, 1243GR, 1325GR, 1345GR, 1354GR, 1425GR, 2435GR.

Ex. 238. 1234, 1235, 1342, 1352, 1354, 1432, 1452, 1453, 1532, 1542, 2143, 2145, 2153, 2354, 2453, 3145, 3214, 3215, 3245, 4125, 4135, 4215, 4235, 4315, 4325, and their reverses.

Ex. 239. 1234, 1235, 1342, 1345, 1352, 1354, 1452, 1542, 2134, 2135, 2345, 2354, 2453, 2543, and their reverses.

Ex. 240. 1234R, 1235, 1342, 1345, 1352, 1354, 1452, 1542, 2145, 2154, 2345, 2354, 3142, 3152, 3452, 3542, 4521.

Ex. 241. 1234, 1235, 1324, 1325, 1423, 1432, 1452, 1453, 1523, 1532, 1542, 1543, 2145, 2154, 2314, 2315, 2345, 2354, 2415, 2514, 3145, 3154, 3214, 3215, 3245, 3254, 3415, 3514, 4235, 4325, and their reverses.

Ex. 242. All tetrads *except* 1243R, 1245R, 1253R, 1254R, 1324R, 1325R, 1423R, 1425R, 1523R, 1524R, 3124R, 3125R, 3245R, 3254R, 3425R, 3524R.

Ex. 243. 1234, 1235, 1243, 1325, 1342, 1345, 1352, 1423, 1425, 1432, 1435, 1452, 1523, 1543, 2134, 2135, 2145, 2154, 2314, 2315, 2345, 2354, 2415, 3124, 3145, 3214, 3215, 3245, 3415, 3425, 3514, 3524, 4125, 4135, 4215, 4235, 4325, and their reverses.

Ex. 244. 1234, 1235, 1342, 1345, 1352, 1354, 1452, 1542, 2134, 2135, 2145, 2154, 2314, 2315, 2345, 2354, 2413, 2453, 2513, 2543, 3145, 3154, 3214, 3215, 4125, 4135, 4215, 4235, 4315, 4325, and their reverses.

Ex. 245. 1234G, 1235G, 1243G, 1245G, 1253G, 1254G, 1425GR, 1435GR, 2314G, 2315G, 2345G, 2354G, 2435GR, 4315G, 5314G.

Ex. 246. 1234GR, 1235G, 1243G, 1245G, 1253GR, 1254GR, 1435GR, 2435GR, 4315G, 4325G.

Ex. 247. All tetrads *except* 1243GR, 1245GR, 1325GR, 1345GR, 1354GR, 2345GR.

Ex. 248. 1234, 1235, 3214, 3215, 4125, 4135, 4215, 4235, 4315, 4325, and their reverses.

Ex. 249. 1234, 1235, 1324, 1325, 1425, 1432, 1435, 1523, 1524, 1534, 2143, 2145, 2315, 2345, 2413, 2415, 2513, 2514, 2543, 3145, 3214, 3254, 3415, 3514, 4235, 4325, and their reverses.

Ex. 250. 1234R, 1235R, 1245R, 1435R, 2345R.

Ex. 251. 1234R, 1235R, 1245R, 2354R, 3415R.

Ex. 252. 1234R, 1235R, 1435R, 1542R, 2345R.

Ex. 253. 1234, 1235, 1245, 1453, 2345R.

Ex. 254. 1234R, 1235, 1435, 2145, 2345, 4521.

Ex. 255. 1234R, 1235R, 1245R, 1254R, 1435R, 2345R, 2354R.

Ex. 256. 1234R, 1235R, 1243R, 1245R, 1253R, 1254R, 1543R, 2143R, 2543R, 3415R, 3425R.

Ex. 257. 1234R, 1235, 1243, 1245, 1253, 1254, 2341, 2345, 2351, 2354, 2431, 2435, 2451, 2453, 2531, 2534, 2541, 2543, 3241, 3251, 3451, 3541, 4231, 4251, 4351, 4521, 4531, 5231, 5241, 5341, 5431.

Ex. 258. All tetrads *except* 1345R, 1354R, 1324R, 2134R.

Ex. 259. 1234G, 1235G, 1243GR, 1245G, 1253G, 1254G, 1324G, 1325G, 1345G, 1354GR, 1425G, 1435GR, 2435GR, 2453G, 2543G.

Ex. 260. 1234R, 1235R, 1345R, 2453R, 4125R.

Ex. 261. 1234, 1235, 1245, 1324, 1325, 1345, 4125, 4135, 4215, 4235, 4315, 4325, and their reverses.

Ex. 262. 1234R, 1235, 1342, 1345, 1423, 1425, 2153, 2453, 4521.

Ex. 263. 1234R, 1235, 1342, 1345, 1352, 1354, 1452, 1542, 2315, 2415, 3142, 3152, 3452, 3542, 4521.

Ex. 264. 1234, 1235, 1245, 1254, 1324, 1325, 1345, 1354, 2143, 2153, 2413, 2453, 2513, 2543, and their reverses.

Ex. 265. 1234R, 1235R, 1345R, 1354R, 2514R, 3524R.

Ex. 266. All tetrads *except* 1245G and 1354G.

Ex. 267. All tetrads *except* 1245GR, 1325GR, 1354GR, 2435GR.

Ex. 268. 1234R, 1235R, 1254R, 1345R, 2354R.

Ex. 269. 1234, 1235, 1253, 1254, 1325, 1342, 1345, 1425R, 1534R, 2134, 2341, 2345, 2351, 2354, 2451, 2531, 2534, 2541, 3125, 3241, 3251, 3421, 3425, 3451, 3541, 4251, 5341.

Ex. 270. 1234R, 1235, 1342, 1345, 1352, 1452, 2145, 2154, 2345, 2354, 3152, 3452, 4521.

Ex. 271. 1234R, 1235R, 1345R, 1425R, 2134R, 2345R, 2354R.

Ex. 272. All tetrads *except* 1243R, 1245R, 1253R, 1254R, 1324R, 1352R, 1354R, 1423R, 1452R, 1453R, 1523R, 1524R, 1542R, 1543R, 2453R, 2513R, 2543R, 3124R, 3154R, 3245R, 3254R, 3524R.

Ex. 273. All tetrads *except* 1243R, 1245R, 1352R, 1354R, 1423R, 1425R, 1532R, 1534R, 2354R, 2534R, 3245R, 3425R.

Ex. 274. All tetrads *except* 1245GR, 1324GR, 2354GR.

Ex. 275. 1234R, 1235, 1254R, 1342, 1345, 1352, 1354, 1452, 1542, 2145, 2345, 3142, 3152, 3452, 3542.

Ex. 276. 1234R, 1235R, 1254R, 1345R, 1354R, 2135R, 2145R, 2345R.

Ex. 277. All tetrads *except* 1243R, 1245R, 1253R, 1423R, 1453R, 1523R, 1524R, 1532R, 1534R, 1543R, 2143R, 2153R, 2154R, 2354R, 2534R, 3245R, 3514R, 3524R.

Ex. 278. 1234, 1235, 1342, 1345, 1352, 1354, 1452, 1542, 2145, 2345, 3142R, 3152R, 3452, 3542.

Ex. 279. 1234R, 1235R, 1345R, 1354R, 1425R, 2135R, 2345R.

Ex. 280. All tetrads *except* 1345G and 2354G.

Ex. 281. 1234R, 1235, 1324, 1352, 1354, 1524, 2145, 2345, 3124, 3524, 4521.

Ex. 282. 1234R, 1235R, 1354R, 1425R, 2345R.

Ex. 283. 1234R, 1235, 1243, 1245, 1342, 1352, 1354, 1435, 1542, 2135, 2145, 2345, 2435, 3142, 3542, 4521.

Ex. 284. 1234R, 1235, 1243, 1245, 1253, 1254R, 1354, 1435, 1534, 2134, 2135, 2143, 2145, 2153, 2154, 2345, 2435, 2534.

Ex. 285. All tetrads *except* 1345R and 2354R.

Ex. 286. All tetrads *except* 1342R, 1345R, 1532R, 1542R, 2134R, 2154R, 2315R, 2345R, 2354R, 4315R.

Ex. 287. 1234, 1235, 1324, 1325, 2145, 2314, 2315, 2345, 3145, 3214, 3215, 3245, 4125, 4135, 4215, 4235, 4315, 4325, and their reverses.

Table of Theorems.

The following table of theorems gives several items of information. The use of the table may best be explained by a concrete illustration.

Line 35, for instance, tells us what can be done with Postulates G, H, 19. Thus:

In the first place, the *italic figures* within parentheses tell us that Postulates G, H, 19 imply Postulates 14, 17, and 18, as proved in Theorem 14.19, Theorem 17.21, and Theorem 18.18 respectively.

In the second place, the *empty brackets* tell us that Postulates G, H, 19 imply also Postulates 15 and 16, but that not all of the Postulates G, H, 19 are required for the proof. (It will be found from line 107 that Postulates 15 and 16 follow from H and 19 alone.)

In the third place, the *dots* tell us that Postulates F, R, 10, 11, 12, 13 are not deducible from Postulates G, H, 19.

Finally, the *Roman numeral* IX at the end of the line shows that the theorems on this line are useful in connection with the "Set IX" of independent postulates. All the theorems required for Set IX (that is, all the theorems required for deducing the remaining postulates from Postulates F, G, H, R, 11, 19) are listed in the lines marked IX. (The lines marked with IX in parentheses contain alternative theorems which may be used in connection with Set IX but are not required for that purpose.)

Line	F′	R′	F	G	H	R	10	11	12	13	14	15	16	17	18	19	Set
1	(F′)	·	F	·	·	·	·	·	·	·	·	·	·	·	·	·	X
2	·	(R′)	·	·	·	R	10	⌒	⌒	⌒	⌒	⌒	⌒	⌒	⌒	⌒	
3	F′	·	(F1)	G	H	·	10	⌒	12	⌒	⌒	⌒	⌒	⌒	⌒	⌒	
4	F′	·	(F2)	G	·	·	10	⌒	⌒	13	⌒	⌒	⌒	⌒	⌒	⌒	
5	F′	·	(F3)	G	·	·	10	·	·	·	14	·	·	·	·	·	
6	F′	·	(F4)	G	·	·	10	·	·	·	·	15	·	·	·	·	
7	F′	·	(F5)	G	·	·	10	·	·	·	⌒	·	16	·	·	·	
8	F′	·	(F6)	G	·	·	10	11	·	·	⌒	·	·	17	·	19	
9	F′	·	(F7)	G	·	·	10	11	·	·	⌒	⌒	⌒	⌒	⌒	⌒	
10	F′	·	(F8)	G	·	·	10	⌒	⌒	⌒	⌒	⌒	⌒	⌒	⌒	⌒	
11	·	R′	·	G	H	(R1)	10	⌒	⌒	⌒	⌒	⌒	⌒	⌒	⌒	⌒	X

Line	F G H R	10	11	12	13	14	15	16	17	18	19	Set
12	F G · R	(10·1)	( )	12	( )	(14·26)	(15·35)	(16·27)	(17·32)	(18·27)	(19·31)	II
13	F G · R	(10·2)	( )	( )	13	(14·27)	(15·36)	(16·28)	(17·33)	(18·28)	(19·32)	III
14	F G · R	(10·3)	11	·	·	14	( )	( )	( )	( )	( )	IV
15	F G · R	(10·4)	11	·	·	( )	15	( )	( )	( )	( )	V
16	F G · R	(10·5)	11	·	·	( )	( )	16	( )	( )	( )	VI
17	F G · R	(10·6)	11	·	·	( )	( )	( )	17	( )	( )	VII
18	F G · R	(10·7)	11	·	·	·	·	·	·	18	·	VIII
19	F G · R	(10·8)	11	·	·	( )	( )	( )	( )	( )	19	IX
20	· G H R	·	·	·	·	14	(15·33)	( )	(17·30)	(18·25)	(19·29)	IV
21	· G H R	·	·	·	·	( )	(15·34)	16	(17·31)	(18·26)	(19·30)	VI
22	· G H R	·	11	(12·10)	(13·9)	14	( )	( )	( )	( )	( )	IV
23	· G H R	·	11	(12·11)	(13·10)	( )	( )	16	( )	( )	( )	VI
24	· G H ·	10	(11·1)	(12·1)	( )	(14·1)	(15·1)	(16·1)	(17·1)	( )	(19·1)	I, X
25	· G H ·	·	11	(12·3)	(13·3)	( )	15	( )	( )	( )	( )	V
26	· G H ·	·	11	(12·4)	( )	( )	( )	( )	17	( )	( )	VII
27	· G H ·	·	11	( )	(13·5)	( )	( )	( )	( )	18	( )	VIII
28	· G H ·	·	11	(12·6)	(13·6)	( )	( )	( )	( )	( )	19	IX
29	· G H ·	·	·	·	·	14	·	(16·16)	·	·	·	IV
30	· G H ·	·	·	·	·	14	15	( )	(17·18)	(18·15)	(19·18)	
31	· G H ·	·	·	·	·	( )	15	16	(17·19)	(18·16)	(19·19)	
32	· G H ·	·	·	·	·	(14·16)	( )	16	·	·	·	VI
33	· G H ·	·	·	·	·	( )	( )	(16·17)	17	(18·17)	(19·20)	VII
34	· G H ·	·	·	·	·	( )	(15·15)	( )	(17·20)	18	(19·21)	VIII
35	· G H ·	·	·	·	·	(14·19)	( )	( )	(17·21)	(18·18)	19	IX
36	· G · R	10	·	·	·	(14·22)	15	(16·23)	17	( )	( )	
37	· G · R	10	·	·	·	(14·23)	15	(16·24)	·	·	19	
38	· G · R	·	11	·	·	14	(15·25)	( )	(17·28)	(18·23)	(19·27)	(IV)
39	· G · R	·	11	·	·	( )	(15·26)	16	(17·29)	(18·24)	(19·28)	(VI)
40	· G · R	·	11	·	·	( )	(15·27)	(16·22)	17	( )	( )	(VII)
41	· G · R	·	11	·	·	(14·21)	(15·28)	( )	( )	( )	19	(IX)
42	· G · R	·	·	·	·	14	·	(16·21)	·	·	·	(IV)
43	· G · R	·	·	·	·	14	(15·29)	( )	17	·	( )	
44	· G · R	·	·	·	·	14	(15·30)	( )	( )	·	19	

Line	F G H R	10	11	12	13	14	15	16	17	18	19	Set
45	· G · R	·	·	·	·	(14·24)	15	(16·25)	17	18	( )	
46	· G · R	·	·	·	·	(14·25)	15	(16·26)	( )	18	19	
47	· G · R	·	·	·	·	(14·20)	·	16	·	·	·	(VI)
48	· G · R	·	·	·	·	( )	(15·31)	16	17	·	( )	
49	· G · R	·	·	·	·	( )	(15·32)	16	( )	·	19	
50	· G · R	·	·	·	·	·	·	·	17	·	(19·26)	VII
51	· G · R	·	·	·	·	·	·	·	(17·27)	·	19	IX
52	· G · ·	10	·	·	·	·	·	·	·	(18·1)	·	I, X
53	· G · ·	10	11	·	·	14	·	·	(17·15)	( )	·	
54	· G · ·	10	11	·	·	14	(15·17)	16	( )	( )	( )	
55	· G · ·	10	11	·	·	·	·	16	·	( )	(19·15)	
56	· G · ·	10	( )	12	( )	(14·2)	(15·2)	(16·2)	(17·2)	( )	(19·2)	
57	· G · ·	10	( )	( )	13	(14·8)	(15·8)	(16·8)	(17·8)	( )	(19·8)	
58	· G · ·	·	11	·	·	14	(15·18)	16	( )	18	( )	
59	· G · ·	·	11	·	·	14	·	·	(17·16)	18	·	
60	· G · ·	·	11	·	·	14	(15·19)	( )	(17·17)	( )	19	
61	· G · ·	·	11	·	·	(14·14)	15	(16·14)	(17·14)	(18·12)	(19·14)	V
62	· G · ·	·	11	·	·	( )	(15·20)	16	17	( )	(19·16)	
63	· G · ·	·	11	·	·	·	·	16	·	18	(19·17)	
64	· G · ·	·	11	·	·	(14·15)	·	·	17	(18·13)	·	VII
65	· G · ·	·	11	·	·	( )	(15·21)	( )	17	( )	19	
66	· G · ·	·	11	·	·	·	·	(16·15)	·	(18·14)	19	IX
67	· G · ·	·	(11·2)	12	(13·2)	·	·	·	·	·	·	II
68	· G · ·	·	( )	12	( )	14	(15·3)	( )	(17·3)	( )	(19·3)	
69	· G · ·	·	( )	12	( )	(14·3)	15	(16·4)	(17·4)	(18·3)	(19·4)	
70	· G · ·	·	( )	12	( )	(14·4)	(15·4)	16	(17·5)	(18·4)	(19·5)	
71	· G · ·	·	( )	12	( )	(14·5)	(15·5)	(16·5)	17	(18·5)	( )	
72	· G · ·	·	( )	12	( )	(14·6)	(15·6)	( )	(17·6)	18	(19·7)	
73	· G · ·	·	( )	12	( )	(14·7)	(15·7)	(16·7)	(17·7)	(18·6)	19	
74	· G · ·	·	(11·3)	(12·2)	13	·	·	·	·	·	·	III
75	· G · ·	·	( )	( )	13	14	( )	(16·9)	( )	(18·7)	(19·9)	
76	· G · ·	·	( )	( )	13	(14·9)	15	(16·10)	(17·10)	(18·8)	(19·10)	
77	· G · ·	·	( )	( )	13	(14·10)	(15·10)	16	(17·11)	(18·9)	(19·11)	

Line	F G H R	10	11	12	13	14	15	16	17	18	19	Set
78	· G · ·	·	( )	( )	13	(14·11)	( )	(16·11)	17	(18·10)	(19·12)	
79	· G · ·	·	( )	( )	13	(14·12)	(15·12)	(16·12)	(17·12)	18	( )	
80	· G · ·	·	( )	( )	13	(14·13)	(15·13)	(16·13)	(17·13)	(18·11)	19	
81	· · H R	10	(11·5)	·	( )	·	·	·	·	·	·	
82	· · H R	10	( )	12	( )	·	(15·44)	16	·	·	(19·37)	
83	· · H R	10	( )	(12·12)	( )	( )	( )	( )	( )	18	( )	
84	· · H R	10	( )	(12·13)	( )	·	( )	( )	·	·	19	
85	· · H R	·	11	( )	(13·11)	( )	(15·37)	( )	(17·34)	18	(19·33)	
86	· · H ·	10	·	·	(13·1)	·	·	·	·	·	·	I, X
87	· · H ·	10	11	(12·7)	( )	·	( )	( )	·	·	19	
88	· · H ·	10	·	12	( )	14	( )	( )	( )	( )	(19·22)	
89	· · H ·	10	·	·	( )	14	(15·22)	·	(17·22)	·	·	
90	· · H ·	10	·	·	( )	14	( )	( )	( )	(18·19)	19	
91	· · H ·	10	·	·	( )	( )	(15·23)	( )	(17·23)	18	(19·23)	
92	· · H ·	·	11	12	(13·7)	·	( )	( )	·	·	19	
93	· · H ·	·	11	(12·8)	13	·	( )	( )	·	·	19	
94	· · H ·	·	11	·	(13·4)	( )	( )	·	17	·	·	VII
95	· · H ·	·	11	(12·9)	( )	( )	( )	( )	17	( )	19	
96	· · H ·	·	11	(12·5)	·	( )	·	( )	·	18	·	VIII
97	· · H ·	·	11	( )	(13·8)	( )	( )	( )	( )	18	19	
98	· · H ·	·	·	12	·	14	( )	( )	(17·26)	( )	19	
99	· · H ·	·	·	12	·	( )	( )	(16·20)	17	(18·22)	( )	
100	· · H ·	·	·	·	13	14	( )	( )	( )	(18·20)	19	
101	· · H ·	·	·	·	13	( )	(15·24)	( )	(17·25)	18	( )	
102	· · H ·	·	·	·	·	(14·17)	(15·14)	·	17	·	·	VII
103	· · H ·	·	·	·	·	( )	( )	( )	17	18	(19·24)	
104	· · H ·	·	·	·	·	( )	( )	( )	17	(18·21)	19	
105	· · H ·	·	·	·	·	(14·18)	·	(16·18)	·	18	·	VIII
106	· · H ·	·	·	·	·	( )	( )	( )	(17·24)	18	19	
107	· · H ·	·	·	·	·	·	(15·16)	(16·19)	·	·	19	IX
108	· · · R	10	11	12	( )	( )	15	( )	(17·38)	18	( )	
109	· · · R	10	11	·	13	(14·29)	15	·	(17·37)	18	( )	
110	· · · R	10	( )	12	13	( )	15	( )	(17·39)	18	( )	

Line	F G H R	10	11	12	13	14	15	16	17	18	19	Set
111	· · · R	10	·	12	·	14	(15·45)	( )	·	( )	(19·39)	
112	· · · R	10	·	12	·	·	15	(16·30)	·	·	(19·38)	
113	· · · R	10	·	12	·	(14·31)	15	( )	·	18	( )	
114	· · · R	10	(11·6)	12	(13·13)	(14·28)	(15·43)	(16·31)	17	(18·30)	( )	
115	· · · R	10	·	12	·	·	·	( )	·	18	(19·40)	
116	· · · R	10	·	12	·	·	·	(16·29)	·	·	19	
117	· · · R	10	·	·	13	(14·30)	( )	·	17	18	( )	
118	· · · R	10	·	·	13	14	( )	·	( )	(18·29)	19	
119	· · · R	10	·	·	13	(14·32)	( )	·	17	(18·31)	19	
120	· · · R	·	11	12	(13·12)	·	·	·	·	·	·	
121	· · · R	·	11	12	( )	14	(15·38)	( )	(17·35)	( )	(19·34)	
122	· · · R	·	11	12	( )	·	(15·39)	·	17	·	( )	
123	· · · R	·	11	12	( )	·	·	( )	·	18	(19·35)	
124	· · · R	·	11	·	·	14	·	·	(17·36)	18	·	
125	· · · R	·	11	·	·	14	(15·41)	·	·	·	19	
126	· · · R	·	11	·	·	·	(15·42)	16	17	·	·	
127	· · · R	·	11	·	·	·	·	16	·	18	(19·36)	
128	· · · R	·	11	·	·	·	(15·40)	·	17	·	19	
129	· · · R	·	(11·4)	12	13	·	·	·	·	·	·	
130	· · · ·	·	·	12	13	14	( )	( )	( )	( )	(19·25)	
131	· · · ·	·	·	12	·	14	·	(16·3)	·	(18·2)	·	
132	· · · ·	·	·	12	·	·	·	·	17	·	(19·6)	
133	· · · ·	·	·	12	·	·	·	(16·6)	·	18	·	
134	· · · ·	·	·	·	13	14	(15·9)	·	(17·9)	·	·	
135	· · · ·	·	·	·	13	·	(15·11)	·	17	·	·	
136	· · · ·	·	·	·	13	·	·	·	·	18	(19·13)	

The ten sets of independent postulates may be exhibited in the same tabular form, as follows:

	F′	R′	F	G	H	R	10	11	12	13	14	15	16	17	18	19
I	( )	( )	F	G	H	R	10	( )	( )	( )	( )	( )	( )	( )	( )	( )
II	( )	( )	F	G	H	R	( )	( )	12	( )	( )	( )	( )	( )	( )	( )
III	( )	( )	F	G	H	R	( )	( )	( )	13	( )	( )	( )	( )	( )	( )
IV	( )	( )	F	G	H	R	( )	11	( )	( )	14	( )	( )	( )	( )	( )
V	( )	( )	F	G	H	R	( )	11	( )	( )	( )	15	( )	( )	( )	( )
VI	( )	( )	F	G	H	R	( )	11	( )	( )	( )	( )	16	( )	( )	( )
VII	( )	( )	F	G	H	R	( )	11	( )	( )	( )	( )	( )	17	( )	( )
VIII	( )	( )	F	G	H	R	( )	11	( )	( )	( )	( )	( )	( )	18	( )
IX	( )	( )	F	G	H	R	( )	11	( )	( )	( )	( )	( )	( )	( )	19
X	F′	R′	( )	G	H	( )	10	( )	( )	( )	( )	( )	( )	( )	( )	( )

Table of Examples.

The following table states which postulates are satisfied by each of the eighty-eight systems (K, R) which were constructed above. The numbering of the examples begins with 200 to avoid confusion with earlier lists.

Either a dot or a dash indicates that the system fails to satisfy the postulate in question. For instance, line 235 of the table shows that Example 235 satisfies Postulates F,′ R,′ F, G, R, 10, 11, 18, but fails to satisfy Postulates H, 12, 13, 14, 15, 16, 17, 19.

The 154 dashes in the table give 154 *theorems on non-deducibility*. For instance, line 235 contains five such theorems on non-deducibility, since it shows that Postulates 14, 15, 16, 17, and 19 cannot be deduced from Postulates F′, R′, F, G, R, 10, 11, 18. The dots in line 235 are to be interpreted as follows: Postulates H, 12, and 13 are not deducible from F′, R′, F, G, R, 10, 11, 18, but in the case of these three postulates stronger examples will be found elsewhere in the table. (Thus, Example 208 is a stronger example for H; and Example 221 is a stronger example for 12 and 13.)

If we wish to find all the theorems on non-deducibility which concern any particular postulate, we have merely to pick out all the dashes which occur in the column corresponding to that postulate.

These 154 theorems on non-deducibility, together with the 267 theorems on deducibility given in the previous table, will be found to be exhaustive; that is, the two tables together enable us to determine at once whether any given postulate of our basic list is or is not deducible from any given combination of the postulates of the list.

For instance, if we ask whether Postulate 15 is deducible from G, 10, 11, 14, 16, 18, the answer is Yes, since line 54 of the table shows that 15 follows from G, 10, 11, 14, 16 even without the use of 18. Again, if we ask whether Postulate 15 is deducible from G, 10, 11, 18, 19, the answer is No, since line 232 of the table shows that 15 is not deducible from G, 10, 11, 18, 19, even with the aid of F′, R′, F, and 16. In a similar manner, every question of this kind can be answered at once, by inspection of the tables.

Ex.	F′	R′	F	G	H	R	10	11	12	13	14	15	16	17	18	19
200	F′	R′	F	G	H	R	10	11	12	13	14	15	16	17	18	19
201	–	R′	–	G	H	R	10	11	12	13	14	15	16	17	18	19
202	F′	–	F	G	H	–	10	11	12	13	14	15	16	17	18	19
203	F′	R′	–	G	H	R	–	11	12	13	14	15	16	17	18	19
204	F′	R′	–	·	H	R	10	11	12	13	14	15	16	17	18	19
205	F′	R′	–	G	·	R	10	11	·	·	·	·	·	·	18	·
206	F′	R′	–	G	·	R	10	·	·	·	·	·	·	17	18	19
207	F′	R′	F	–	H	R	10	11	12	13	14	15	16	17	18	19
208	F′	R′	F	G	–	R	10	11	12	13	14	15	16	17	18	19
209	F′	R′	F	·	H	–	10	11	12	13	14	15	16	17	18	19
210	F′	R′	F	G	·	–	10	11	12	13	14	15	16	17	18	19
211	F′	R′	F	G	H	–	–	11	12	13	14	15	16	17	18	19
212	F′	R′	F	·	H	R	–	11	12	13	14	15	16	17	18	19
213	F′	R′	F	G	H	R	–	–	–	–	14	15	16	17	18	19
214	F′	R′	F	G	H	R	–	11	–	–	–	–	–	–	–	–
215	F′	R′	F	G	·	R	10	–	·	·	14	15	16	17	18	19
216	F′	R′	F	·	·	R	10	–	12	–	14	15	16	–	18	19
217	F′	R′	F	·	·	R	10	–	·	13	14	15	16	17	18	19
218	F′	R′	F	·	H	R	·	–	–	13	14	15	16	17	18	19
219	F′	R′	F	·	H	R	·	–	12	–	14	15	16	17	18	19
220	F′	R′	F	·	H	·	10	–	12	13	14	15	16	17	18	19
221	F′	R′	F	G	·	R	10	11	–	–	14	15	16	17	18	19
222	F′	R′	F	G	H	·	·	11	–	–	14	–	16	–	–	–
223	F′	R′	F	·	·	R	10	11	–	13	14	15	16	17	18	19
224	F′	R′	F	·	H	R	10	11	–	13	14	15	16	17	–	–
225	F′	R′	F	·	H	·	10	·	–	13	14	15	16	17	18	19
226	F′	R′	F	·	H	R	·	11	–	–	14	15	16	–	–	19
227	F′	R′	F	·	·	·	10	11	12	–	14	15	16	17	18	19
228	F′	R′	F	·	H	·	·	11	12	–	14	15	16	–	18	–
229	F′	R′	F	G	H	R	·	·	·	·	–	15	–	–	–	–
230	F′	R′	·	G	H	R	·	11	12	13	–	–	–	–	–	–
231	F′	R′	F	G	H	·	·	11	12	13	–	–	–	–	–	–
232	F′	R′	F	G	·	·	10	11	·	·	–	–	16	–	18	19
233	F′	R′	F	G	·	·	10	·	·	·	–	15	16	17	18	19
234	F′	R′	F	G	·	R	10	·	·	·	–	–	–	17	18	19
235	F′	R′	F	G	·	R	10	11	·	·	–	–	–	–	18	–
236	F′	R′	F	G	·	R	10	·	·	·	–	15	–	·	18	·
237	F′	R′	F	G	·	R	·	·	·	·	–	15	–	17	·	19
238	F′	R′	F	·	H	R	10	11	12	13	–	15	16	–	–	19
239	F′	R′	F	·	·	R	·	11	12	13	–	15	16	17	18	19
240	F′	R′	F	·	·	·	10	11	12	13	–	15	16	17	18	19
241	F′	R′	F	·	·	R	10	11	12	13	–	–	16	–	18	19
242	F′	R′	F	·	·	R	10	11	·	·	–	15	16	17	18	19
243	F′	R′	F	·	·	R	10	·	·	13	–	15	16	–	18	19
244	F′	R′	F	·	·	R	10	11	·	13	–	15	16	17	·	·
245	F′	R′	F	G	·	·	10	·	·	·	14	–	16	17	18	19
246	F′	R′	F	G	·	·	10	11	·	·	14	–	–	17	18	–
247	F′	R′	F	G	·	R	10	·	·	·	14	–	16	·	18	·
248	F′	R′	F	·	H	R	10	11	12	13	·	–	–	·	·	–
249	F′	R′	F	·	H	R	10	11	·	13	·	–	16	·	·	·
250	F′	R′	F	·	H	R	·	·	12	·	14	–	16	·	18	·
251	F′	R′	F	·	H	R	·	11	·	·	14	–	16	·	·	·

Ex.	F′	R′	F	G	H	R	10	11	12	13	14	15	16	17	18	19
252	F′	R′	F	·	H	R	·	11	12	13	·	–	16	·	·	·
253	F′	R′	F	·	H	·	·	11	12	·	14	–	16	·	18	·
254	F′	R′	F	·	H	·	10	11	12	13	·	–	16	·	·	·
255	F′	R′	F	·	·	R	·	·	12	·	14	–	16	17	18	19
256	F′	R′	F	·	·	R	10	11	·	·	14	–	·	17	18	·
257	F′	R′	F	·	·	·	10	11	12	·	14	–	16	17	18	19
258	F′	R′	F	·	·	R	10	·	·	·	14	–	16	17	18	19
259	F′	R′	F	G	·	·	10	·	·	·	14	15	–	17	18	19
260	F′	R′	F	·	H	R	·	11	12	13	·	15	–	·	·	·
261	F′	R′	F	·	H	R	10	11	·	13	14	15	–	17	·	·
262	F′	R′	F	·	H	·	10	11	12	13	·	15	–	·	·	·
263	F′	R′	F	·	·	·	10	11	12	13	·	15	–	17	·	19
264	F′	R′	F	·	·	R	10	11	·	13	14	15	–	17	18	19
265	F′	R′	F	·	·	R	·	11	12	13	·	15	–	17	·	19
266	F′	R′	F	G	·	·	10	·	·	·	14	15	16	–	18	19
267	F′	R′	F	G	·	R	10	·	·	·	14	15	16	–	18	–
268	F′	R′	F	·	H	R	·	·	12	·	14	15	16	–	18	–
269	F′	R′	F	·	·	·	10	11	12	·	14	15	16	–	18	19
270	F′	R′	F	·	·	·	10	11	12	13	·	15	16	–	18	19
271	F′	R′	F	·	·	R	·	11	12	13	·	15	16	–	18	19
272	F′	R′	F	·	·	R	10	11	·	·	·	15	16	–	18	19
273	F′	R′	F	·	·	R	10	11	·	·	14	15	16	–	·	19
274	F′	R′	F	G	·	R	·	·	·	·	14	15	16	17	–	19
275	F′	R′	F	·	·	·	10	11	·	13	14	15	16	17	–	19
276	F′	R′	F	·	·	R	·	11	·	13	14	15	16	17	–	19
277	F′	R′	F	·	·	R	10	11	·	·	14	15	16	17	–	19
278	F′	R′	F	·	·	·	10	11	12	13	·	15	16	17	–	19
279	F′	R′	F	·	·	R	·	11	12	13	·	15	16	17	–	19
280	F′	R′	F	G	·	·	10	·	·	·	14	15	16	17	18	–
281	F′	R′	F	·	H	·	10	11	12	13	·	15	16	·	·	–
282	F′	R′	F	·	H	R	·	11	12	13	·	15	16	·	·	–
283	F′	R′	F	·	·	·	10	11	12	·	14	15	16	·	18	–
284	F′	R′	F	·	·	·	10	11	·	·	14	15	16	17	18	–
285	F′	R′	F	·	·	R	10	·	·	·	14	15	16	17	18	–
286	F′	R′	F	·	·	R	10	11	·	·	14	15	·	17	18	–
287	F′	R′	F	·	·	R	10	11	12	13	·	·	16	·	·	–

These two tables, which summarize the conclusions of the paper, are complete, and free from redundancies.